Schrodinger Functional

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The Schrodinger functional in Quantum Field Theory evolves according to the time-dependent Schrodinger equation with a Hamiltonian obtained from the Klein-Gordon equation by the standard prescription, but replacing derivatives with functional derivatives. This Hamiltonian is given in Hatfield (10.11)

$$\hat{H} = \frac{1}{2}\int d^3x' \left(-\frac{\delta^2}{\delta\phi^2(\mathbf{x}')} + |\boldsymbol{\nabla}'\phi(\mathbf{x}')|^2 + m^2\phi^2(\mathbf{x}') \right)$$

Using this Hamiltonian ensures that the field states evolve according to the Klein-Gordon equation, just as using the Hamiltonian for a particle in a field ensures that the particle position states evolve according to Newton's equations. We already showed the case of particles using the Ehrenfest theorem. We will now show the case for fields using the Ehrenfest theorem again, which still applies because it works for any system that obeys the Schrodinger equation.
In the particle case we applied the Ehrenfest theorem to the conjugate momentum $p = -i\hbar\frac{\partial}{\partial x}$. We will now do the same thing with the conjugate momentum $\pi(\mathbf{x}) = -i\frac{\delta}{\delta\phi(\mathbf{x})}$.[1] First we compute the commutators with individual terms in the Hamiltonian density.

$$\left[\frac{\delta}{\delta\phi(\mathbf{x})}, \frac{\delta^2}{\delta\phi^2(\mathbf{x}')}\right] = 0$$

$$\left[\frac{\delta}{\delta\phi(\mathbf{x})}, |\boldsymbol{\nabla}'\phi(\mathbf{x}')|^2\right] = \left[\frac{\delta}{\delta\phi(\mathbf{x})}, \left(\frac{\partial\phi(\mathbf{x}')}{\partial x'}\right)^2 + \left(\frac{\partial\phi(\mathbf{x}')}{\partial y'}\right)^2 + \left(\frac{\partial\phi(\mathbf{x}')}{\partial z'}\right)^2\right]$$

If we just look at one piece,

$$\left[\frac{\delta}{\delta\phi(\mathbf{x})}, \left(\frac{\partial\phi(\mathbf{x}')}{\partial x'}\right)^2\right]\psi = \frac{\delta}{\delta\phi(\mathbf{x})}\left(\left(\frac{\partial\phi(\mathbf{x}')}{\partial x'}\right)^2\psi\right) - \left(\frac{\partial\phi(\mathbf{x}')}{\partial x'}\right)^2\frac{\delta\psi}{\delta\phi(\mathbf{x})}$$

$$= \frac{\delta}{\delta\phi(\mathbf{x})}\left(\left(\frac{\partial\phi(\mathbf{x}')}{\partial x'}\right)^2\right)\psi = 2\frac{\partial\phi(\mathbf{x}')}{\partial x'}\frac{\delta}{\delta\phi(\mathbf{x})}\left(\frac{\partial\phi(\mathbf{x}')}{\partial x'}\right)\psi$$

$$= 2\frac{\partial\phi(\mathbf{x}')}{\partial x'}\frac{\partial}{\partial x'}\left(\frac{\delta\phi(\mathbf{x}')}{\delta\phi(\mathbf{x})}\right)\psi = 2\frac{\partial\phi(\mathbf{x}')}{\partial x'}\frac{\partial}{\partial x'}\left(\delta(\mathbf{x}'-\mathbf{x})\right)\psi$$

Now we can add all three pieces back together,

$$\left[\frac{\delta}{\delta\phi(\mathbf{x})}, |\boldsymbol{\nabla}'\phi(\mathbf{x}')|^2\right] = 2\boldsymbol{\nabla}'\phi(\mathbf{x}')\cdot\boldsymbol{\nabla}'\delta(\mathbf{x}'-\mathbf{x})$$

$$\left[\frac{\delta}{\delta\phi(\mathbf{x})}, \phi^2(\mathbf{x}')\right]\psi = \frac{\delta}{\delta\phi(\mathbf{x})}\left(\phi^2(\mathbf{x}')\psi\right) - \phi^2(\mathbf{x}')\frac{\delta\psi}{\delta\phi(\mathbf{x})} = \frac{\delta}{\delta\phi(\mathbf{x})}\left(\phi^2(\mathbf{x}')\right)\psi$$

$$\left[\frac{\delta}{\delta\phi(\mathbf{x})}, \phi^2(\mathbf{x}')\right] = 2\phi(\mathbf{x}')\frac{\delta\phi(\mathbf{x}')}{\delta\phi(\mathbf{x})} = 2\phi(\mathbf{x}')\delta(\mathbf{x}'-\mathbf{x})$$

Now we are ready to compute the commutator of the conjugate momentum with the Hamiltonian.

$$[\pi(\mathbf{x}), \hat{H}] = -i\int d^3x' \left[\frac{\delta}{\delta\phi(\mathbf{x})}, \hat{\mathcal{H}}\right]$$

$$= -i\int d^3x'\frac{1}{2}\left(2\boldsymbol{\nabla}'\phi(\mathbf{x}')\cdot\boldsymbol{\nabla}'\delta(\mathbf{x}'-\mathbf{x}) + 2m^2\phi(\mathbf{x}')\delta(\mathbf{x}'-\mathbf{x})\right)$$

[1] See Hatfield (10.9)

$$= -i \int d^3x' \left(\boldsymbol{\nabla}'\phi(\mathbf{x}') \cdot \boldsymbol{\nabla}'\delta(\mathbf{x}' - \mathbf{x}) \right) - im^2\phi(\mathbf{x})$$

Using integration by parts,

$$= -i \int d^3x' \left(-\nabla'^2\phi(\mathbf{x}')\delta(\mathbf{x}' - \mathbf{x}) \right) - im^2\phi(\mathbf{x})$$

$$= i\nabla^2\phi(\mathbf{x}) - im^2\phi(\mathbf{x})$$

So by the Ehrenfest Theorem,

$$\frac{d}{dt} \langle \Phi(t)| \pi(\mathbf{x}) |\Phi(t)\rangle = \left\langle \Phi(t) \right| \nabla^2\phi(\mathbf{x}) - m^2\phi(\mathbf{x}) \left| \Phi(t) \right\rangle$$

If we now convert this to the Heisenberg picture, $\phi(\mathbf{x}) \to \phi(\mathbf{x}, t)$, and $\pi(\mathbf{x}) \to \frac{\partial}{\partial t}\phi(\mathbf{x}, t)$

$$\frac{d}{dt} \left\langle \Phi \left| \frac{\partial}{\partial t}\phi(\mathbf{x}, t) \right| \Phi \right\rangle = \left\langle \Phi \right| \nabla^2\phi(\mathbf{x}, t) - m^2\phi(\mathbf{x}, t) \left| \Phi \right\rangle$$

But since the states are time independent in the Schrodinger picture,

$$\frac{d}{dt} \left\langle \Phi \left| \frac{\partial}{\partial t}\phi(\mathbf{x}, t) \right| \Phi \right\rangle = \left\langle \Phi \left| \frac{\partial^2}{\partial t^2}\phi(\mathbf{x}, t) \right| \Phi \right\rangle$$

Therefore

$$\left\langle \Phi \left| \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \phi(\mathbf{x}, t) \right| \Phi \right\rangle = 0$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \langle \Phi| \phi(\mathbf{x}, t) |\Phi\rangle = 0$$

This is the statement that the expectation value of the field obeys the Klein-Gordon equation.