Ehrenfest's Theorem

Chris Clark February 1, 2008

1 The Ehrenfest Theorem

The Ehrenfest Theorem states

$$\frac{d}{dt}\langle \hat{A}\rangle = \frac{1}{i\hbar}\langle[\hat{A},\hat{H}]\rangle + \left\langle \frac{\partial \hat{A}}{\partial t}\right\rangle$$

This can be proved directly from the Schrodinger equation.

$$\begin{aligned}\frac{d}{dt}\langle \hat{A}\rangle &= \frac{d}{dt}\int \psi^* \hat{A}\psi \ d^3x \\ &= \int \frac{\partial}{\partial t}\left(\psi^*\hat{A}\psi\right) \ d^3x \\ &= \int \left[\frac{\partial \psi^*}{\partial t}\hat{A}\psi + \psi^*\frac{\partial \hat{A}}{\partial t}\psi + \psi^*\hat{A}\frac{\partial \psi}{\partial t}\right] \ d^3x\end{aligned}$$

Now we use the Schrodinger equation

$$\hat{H}\psi = i\hbar\frac{\partial \psi}{\partial t} \Rightarrow \frac{\partial \psi}{\partial t} = \frac{1}{i\hbar}\hat{H}\psi$$

and the Hermitian conjugate

$$\frac{\partial \psi^\dagger}{\partial t} = -\frac{1}{i\hbar}(\hat{H}\psi)^\dagger \Rightarrow \frac{\partial \psi^*}{\partial t} = -\frac{1}{i\hbar}\psi^*\hat{H}$$

since $\hat{H}^\dagger = \hat{H}$ and $\psi^\dagger = \psi^*$.

$$\begin{aligned}\frac{d}{dt}\langle \hat{A}\rangle &= \left\langle \frac{\partial \hat{A}}{\partial t}\right\rangle + \int \left[-\frac{1}{i\hbar}\psi^*\hat{H}\hat{A}\psi + \psi^*\hat{A}\frac{1}{i\hbar}\psi\right] \ d^3x \\ &= \left\langle \frac{\partial \hat{A}}{\partial t}\right\rangle + \frac{1}{i\hbar}\left\langle [\hat{A},\hat{H}]\right\rangle\end{aligned}$$

2 Force Equations

We can use the Ehrenfest theorem to derive the fact that Schrodinger's equation replaces Newton's laws and the Lorentz force law because it implies

$$\frac{d}{dt}\langle \hat{\mathbf{p}}\rangle = -\langle U(\mathbf{x})\rangle$$

We start by computing

$$[\hat{\mathbf{p}},\hat{H}] = \left[\hat{\mathbf{p}}, \frac{\hat{\mathbf{p}}^2}{2m} + U\right] = [\hat{\mathbf{p}}, U] = -i\hbar[\boldsymbol{\nabla}, U]$$

Therefore, by the Ehrenfest theorem,

$$\begin{aligned}\frac{d}{dt}\langle \hat{\mathbf{p}}\rangle &= \frac{1}{i\hbar}\int \psi^*(-i\hbar)(\boldsymbol{\nabla}U - U\boldsymbol{\nabla})\psi \ d^3x \\ &= -\int \psi^*\boldsymbol{\nabla}(U\psi) - \psi^*U\boldsymbol{\nabla}\psi \ d^3x \\ &= -\int \psi^*\boldsymbol{\nabla}(U)\psi \ d^3x = -\langle \boldsymbol{\nabla}U\rangle\end{aligned}$$