Calculus of Units

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Definition 1. $[x]$ represents the units of the expression x.

Axiom 1. $[xy] = [x][y]$ because units are always multiplicative.

Axiom 2. $[1/x] = 1/[x]$ because units are inverted when an expression is inverted.

Axiom 3. $[x \pm y] = [x]$ if $[x] = [y]$, and $[x \pm y]$ is undefined if $[x] \neq [y]$ because units are additive if they are the same, but it is not possible to add different units.

Axiom 4. We will assume that the units of a function do not depend on the magnitude of the parameter, though they can depend on the units of the parameter.

Theorem 5. $[\lim_{x \to a} f(x)] = [f(x)]$

Proof. The left side is the units of the f evaluated at some value and the right side is the units of the f evaluated at an arbitrary value, but by Axiom 4, these units must be the same. □

Theorem 6. $[\frac{df(x)}{dx}] = \frac{[f(x)]}{[x]}$

Proof. $[\frac{df(x)}{dx}] = [\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}] = [\frac{f(x+h)-f(x)}{h}]$
$= [f(x+h) - f(x)]/[h] = [f(x)]/[h] = [f(x)]/[x]$ since we must assign h the same units as x in order for the quantity $x + h$ to make any sense. □

Theorem 7. $[\sum_n f(n)] = [f(n)]$

Proof. This follows immediately from repeated application of Axiom 3 along with the fact that $[f(n)] = [f(n+1)]$ by Axiom 4. □

Theorem 8. $[\int_a^b f(x)dx] = [f(x)][x]$

Proof. $[\int_a^b f(x)dx] = [\lim_{dx \to 0} \sum_{n=0}^{floor((b-a)/dx)} f(a + n * dx)dx]$
$= [\sum_{n=0}^{floor((b-a)/dx)} f(a + n * dx)dx]$
$= [f(a + n * dx)dx] = [f(a + n * dx)][dx] = [f(x)][x]$ since $[x] = [dx]$. □

Theorem 9. $[\delta(x)] = 1/[x]$

Proof. $[\int_{-\infty}^{\infty} f(x)\delta(x)dx] = [f(0)] \Rightarrow [f(x)\delta(x)][x] = [f(0)]$
$\Rightarrow [f(x)][\delta(x)][x] = [f(0)] \Rightarrow [\delta(x)][x] = 1 \Rightarrow [\delta(x)] = 1/[x]$. □

Theorem 10. $[\delta(g(x))] = 1/[g(x)]$

Proof. $[\int_{-\infty}^{\infty} f(x)\delta(g(x))dx] = [|\frac{dg(x)}{dx}|^{-1}_{f(x)=0} \int_{-\infty}^{\infty} f(x)\delta(g(x))d(g(x))]$
$\Rightarrow [f(x)][\delta(g(x))][x] = [\frac{dg(x)}{dx}]^{-1}[f(c)|_{g(c)=0}]$
$\Rightarrow [\delta(g(x))][x] = [x]/[g(x)] \Rightarrow [\delta(g(x))] = 1/[g(x)]$. □