The Total Derivative

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1 Function Notation

The notation $f(x)$ refers to the value of the function f when applied to the argument x. We can also talk about properties of a function f without applying it to a specific argument. A function f has one or more parameters which correspond to the arguments that the function takes. Often it is not necessary to give names to these parameters, but in some cases like taking derivatives of multi-parameter functions, it becomes necessary to refer to the parameters somehow and naming them is a convenient solution.

The common approach to naming function parameters is with an abuse of notation, using the same notation for the value of f at x, "$f(x)$", to mean "a function f with parameter x".

This can lead to some confusion, so here we will use the following notation:

- $f(x)$ is the value of the function f at x
- $f(\underline{x})$ is a function f of parameter x

Underlined variables can be thought of as annotations with no semantic meaning since $f(\underline{x}) = f$.

A function's parameter names are arbitrary and can be chosen freely without affecting the value of any applications. It doesn't matter if argument names match with parameter names; $f(y)$ is a valid application of a function $f(\underline{x})$ if y is a defined value.

2 Derivative Notation

In Leibniz notation, the derivative of a single-parameter function $f(\underline{x})$ is written as

$$\frac{df}{dx}$$

The x in the notation represents the name of the parameter of the function f. For single-parameter functions, having the parameter name in the notation is redundant since there is only one parameter, and in fact it is somewhat strange that it appears in the notation because the parameter name is totally arbitrary and sometimes isn't even specified. However, for multi-parameter functions we need a way of specifying which parameter to take the derivative with respect to, and showing the parameter allows us to do this. By including the parameter name here even for the single-parameter case, we can make the notation more consistent across all functions arities.

The derivative of a multi-parameter function $f(..., \underline{x}, ...)$ with respect to the parameter named x is written as

$$\frac{\partial f}{\partial x}$$

So in the absence of any abuse of notation:

- The Leibniz notation for a derivative of a single-parameter function must use the d prefix.
- The Leibniz notation for a derivative of a multi-parameter function must use the ∂ prefix.
- The name in the bottom of the Leibniz notation for a derivative must be the name of a parameter to the function in the top.

3 The Chain Rule

The chain rule says that if we have functions $f(\underline{x})$, $g(\underline{y})$, and $h(\underline{x})$ where $h(x) = g(f(x))$ then

$$\frac{dh}{dx}(x) = \frac{dg}{dy}(f(x))\frac{df}{dx}(x)$$

Note that we could have chosen the name of the parameter of g to be "f" so that it more closely resembled the argument $f(x)$, which would also reduce the total number of names in use. This naming is potentially confusing because it gives the same name to two different things, but it isn't technically ambiguous because the the two names are effectively in different namespaces: the names on the bottom of derivatives are always parameter names and all other names are value names. With this renaming we have:

$$\frac{dh}{dx}(x) = \frac{dg}{df}(f(x))\frac{df}{dx}(x)$$

With an abuse of notation, we can suppress the arguments and re-insert them when necessary (if we can safely remember what they should be):

$$\frac{dh}{dx} = \frac{dg}{df}\frac{df}{dx}$$

4 The Total Derivative

The total derivative is a generalization of the chain rule.

Say we have functions $x(\underline{t})$, $y(\underline{t})$, and $f(\underline{u}, \underline{v})$. Then if we define a function $\tilde{f}(\underline{t})$ by $\tilde{f}(t) = f(x(t), y(t))$, the generalization of the chain rule tells us that

$$\frac{d\tilde{f}}{dt}(t) = \frac{\partial f}{\partial u}(x(t), y(t))\frac{dx}{dt}(t) + \frac{\partial f}{\partial v}(x(t), y(t))\frac{dy}{dt}(t)$$

Using the same conventions of renaming the parameters $f(\underline{u}, \underline{v}) \longrightarrow f(\underline{x}, \underline{y})$ and suppressing arguments:

$$\frac{d\tilde{f}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

It is a common abuse of notation to drop the tilde on f and write this as $\frac{df}{dt}$.[1] We can almost think of this as an extension of the d-prefix Leibniz notation because it isn't currently defined for multi-parameter functions. Since we know f is a multi-parameter function, if we see it inside a d-prefix derivative, this indicates that we have to implicitly convert it to a single-parameter function. The problem is that the functions $x(\underline{t})$ and $y(\underline{t})$ needed to do the conversion are not explicit in the notation $\frac{df}{dt}$, which means that the notation is ambiguous.

Let's say we want to compare $\tilde{f}(\underline{t})$ with a new function $\tilde{f}'(\underline{t})$ given by $\tilde{f}'(t) = f(x'(t), y'(t))$ for some additional functions $x'(\underline{t})$ and $y'(\underline{t})$. Because f occurs in both $\tilde{f}$ and $\tilde{f}'$, and a single function should only have a single set of parameter names, we keep f as $f(\underline{x}, \underline{y})$. In this case the total derivative is

$$\frac{d\tilde{f}'}{dt}(t) = \frac{\partial f}{\partial x}(x'(t), y'(t))\frac{dx'}{dt}(t) + \frac{\partial f}{\partial y}(x'(t), y'(t))\frac{dy'}{dt}(t)$$

[1] See The Calculus of Several Variables by Robert C. Rogers page 107.

Notice that now we have to be more careful about the distinction between parameter names and argument names because we have two sets of argument names in scope (primed and unprimed) for the same function, which only has one set of parameter names.

If we had chosen to rename the parameters of f to $f(\underline{x}', \underline{y}')$, this equation for $\frac{d\tilde{f}'}{dt}$ would look more like the usual chain rule form where the tops and bottoms cancel, but then the corresponding equation for $\frac{d\tilde{f}}{dt}$ wouldn't have the usual form. If we use different parameter names for each case, we could make both equations have the usual form, but then we would have to be keep track of the translations $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}$ and $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y'}$ because in this case primed and unprimed parameters both refer to the same parameter of f. Maintaining f as $f(\underline{x}, \underline{y})$ throughout seems preferable.