Pythagorean Theorem

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Notation: (a, b) will denote the vector with x component a and y component b in the Cartesian plane. $|(a, b)|$ will denote the length of the vector (a, b). $(a, b) + (c, d)$ will denote the vector sum $(a + c, b + d)$.

Theorem 0.1. $|(a, b)|^2 = a^2 + b^2$

Proof. Let $(a, b)'$ denote the vector created by rotating (a, b) by $90°$ counterclockwise. Then $(1, 0)' = (0, 1)$ and $(0, 1)' = (-1, 0)$. Now $(a, b)' = [(a, 0) + (0, b)]' = (a, 0)' + (0, b)'$ since the rotation of a vector sum is the vector sum of the rotated vectors. So $(a, b)' = a(1, 0)' + b(0, 1)' = a(0, 1) + b(-1, 0) = (0, a) + (-b, 0) = (-b, a)$.

When calculating a length of a sum of two perpendicular vectors, the vectors can be replaced by any other perpendicular vectors of the same length without affecting the result since length is invariant under rotation. Let $l = |(a, b)|$ and consider the general case $|c(a, b) + d(a, b)'| = |c * l(1, 0) + d * l(1, 0)'| = l|c(1, 0) + d(0, 1)| = |(a, b)||(c, d)|$.

Therefore, $|(a, b)|^2 = |(a, b)||(a, -b)| = |a(a, b) + (-b)(a, b)'| =$
$|(a^2, ab) + (-b)(-b, a)| = |(a^2, ab) + (b^2, -ab)| = |(a^2 + b^2, 0)| = a^2 + b^2$. □