Lagrange's Equation

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1 Calculus of Variations

The **variation** of a function f is

$$\delta f[x(t)] = \lim_{\epsilon \to 0} \frac{f[x(t) + \epsilon\eta(t)] - f[x(t)]}{\epsilon} \tag{1}$$

where $\eta(t)$ is an arbitrary function subject to the constraint that it vanishes at the endpoints of the interval under consideration. First we notice that for f being the identity function we have

$$\delta x(t) = \lim_{\epsilon \to 0} \frac{x(t) + \epsilon\eta(t) - x(t)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\epsilon\eta(t)}{\epsilon} = \eta(t), \tag{2}$$

so we are able to write δx and $\eta(t)$ interchangeably. Now, we can always write the first term in the numerator of (1) as a power series in ϵ, so

$$\delta f[x(t)] = \lim_{\epsilon \to 0} \frac{f[x(t)] + \epsilon\eta(t) f'[x(t)] + O(\epsilon^2) - f[x(t)]}{\epsilon} = f'[x(t)]\delta x \tag{3}$$

This is basically a form of chain rule for variations. We can also derive the following simple rules:

$$\delta(cf[x(t)]) = c\delta f[x(t)] \tag{4}$$

$$\delta(f[x(t)] \pm g[x(t)]) = \delta f[x(t)] \pm \delta g[x(t)] \tag{5}$$

$$\delta(f[g[x(t)]]) = f'[g[x(t)]]\delta(g[x(t)]) \tag{6}$$

$$\delta\left(\frac{dx(t)}{dt}\right) = \frac{d}{dt}(\delta x) \tag{7}$$

2 Principle of Least Action

The **Lagrangian** of a system, written L, is defined to be the difference of the potential and kinetic energies of that system. It is a functional of the positions and velocities of the particles in the system. The **action** of the system in a time interval (t_1, t_2) is defined to be the integral of the Lagrangian with respect to time over the interval. The **Principle of Least Action** says that the actual trajectories and velocities of the particles in the real world will make the action extremal. This principle is equivalent to Newton's laws of motion, but it will provide us with new problem solving methods.

Theorem 2.1. *(The Principle of Least Action)* $\delta \int_{t_1}^{t_2} L\, dt = 0$.

Proof. We will only provide a proof for the special case of one particle in one dimension, but the general proof is a straightforward elaboration. We start with Newton's law $F = m\frac{d^2x}{dt^2}$ and the definition of potential energy $F = -\frac{dV}{dx}$:

$$\Rightarrow -m\frac{d^2x}{dt^2} - \frac{dV}{dx} = 0$$

Now let $\delta x = \eta(t)$ be an arbitrary function that vanishes at the endpoints of the interval (t_1, t_2).[1] Integrating any expression that is equal to zero always yields zero, so:

$$\Rightarrow \int_{t_1}^{t_2} \left(-m\frac{d^2x}{dt^2} - \frac{dV}{dx}\right) \delta x\, dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left(-m\frac{d^2x}{dt^2}\delta x - \frac{dV}{dx}\delta x\right)\, dt = 0$$

Here we integrate by parts on the first term and use the constraint that δx vanishes at the endpoints so that the boundary term is zero.

$$\Rightarrow -m\frac{dx}{dt}\delta x|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(m\frac{dx}{dt}\frac{d}{dt}(\delta x) - \frac{dV}{dx}\delta x\right)\, dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left(m\frac{dx}{dt}\delta\left(\frac{dx}{dt}\right) - \frac{dV}{dx}\delta x\right)\, dt = 0$$

Finally, we use the chain rule of variations backwards to pull the variation symbol out front.

$$\Rightarrow \int_{t_1}^{t_2} \left(\frac{m}{2}\,\delta\left(\left(\frac{dx}{dt}\right)^2\right) - \delta V(x)\right)\, dt = 0$$

$$\Rightarrow \delta \int_{t_1}^{t_2} \left(\frac{m}{2}\dot{x}^2 - V(x)\right)\, dt = 0$$

$$\Rightarrow \delta \int_{t_1}^{t_2} L[x(t), \dot{x}(t)]\, dt = 0$$

□

[1] There is no loss of generality with this assumption because this is built into the definition of variations.

3 Euler-Lagrange Equation

The **Euler-Lagrange Equation** is a mathematical result that converts an equation of the form $\delta \int f dt = 0$ into a differential equation in terms of f. This result can then be directly applied to the principle of least action to yield the Lagrange equations of motion.

Theorem 3.1. *(Euler-Lagrange Equation)*

$$\delta \int f[x(t), \dot{x}(t)]\ dt = 0 \Rightarrow \frac{d}{dt}\frac{df}{d\dot{x}} - \frac{df}{dx} = 0$$

Proof.

$$\delta \int_{t_1}^{t_2} f[x(t), \dot{x}(t)]\ dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \delta f[x(t), \dot{x}(t)]\ dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left(\frac{df}{dx}\delta x + \frac{df}{d\dot{x}}\delta\dot{x}\right)\ dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left(\frac{df}{dx}\delta x + \frac{df}{d\dot{x}}\frac{d}{dt}(\delta x)\right)\ dt = 0$$

$$\Rightarrow \frac{df}{d\dot{x}}\delta x|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{df}{dx}\delta x - \frac{d}{dt}\frac{df}{d\dot{x}}\delta x\right)\ dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left(\frac{df}{dx} - \frac{d}{dt}\frac{df}{d\dot{x}}\right)\delta x\ dt = 0$$

Now for this to be true for an arbitrary function δx, the other factor of the integrand must always be zero.

$$\Rightarrow \frac{d}{dt}\frac{df}{d\dot{x}} - \frac{df}{dx} = 0$$

□

The hypothesis of this theorem is exactly what we have for the action integral, so we immediately obtain Lagrange's equation of motion

$$\Rightarrow \frac{d}{dt}\frac{dL}{d\dot{x}} - \frac{dL}{dx} = 0$$

Let's go back and take a look at how we got to this point. To obtain the least action principle, we basically used the calculus of variations backwards (zero valued expression → integral with zero variation). Then we took that result and applied the calculus of variations in the normal forward direction (integral with zero variation → zero valued expression). So essentially we are back to where we started. If we plug the Lagrangian back in,

$$\frac{d}{dt}\frac{d}{d\dot{x}}\left(\frac{m}{2}\dot{x}^2\right) - \frac{d}{dx}(-V) = 0$$
$$\Rightarrow m\frac{d}{dt}\dot{x} + \frac{dV}{dx} = 0$$
$$\Rightarrow m\frac{d^2x}{dt^2} = -\frac{dV}{dx}$$

and we get Newton's equation back. Then what have we gained? Well, the Lagrangian is a scalar function, so it is independent of coordinate transformations. This makes it easier to solve problems in non-rectangular coordinate systems. Furthermore, the Lagrangian formalism makes it straightforward to apply constraints on the motion, which can be extremely complicated to apply in Newton's formalism.

4 References

The Feynman Lectures on Physics Volume II by Feynman, Leighton, and Sands
Chapter 19

Classical Dynamics of Particles and Systems by Marion and Thornton
Chapter 7