7. *Statistical Mechanics and Thermodynamics* (Spring 2006)

(a) In the case of a set of non-relativistic, noninteracting spin-1/2 fermions confined to two dimensions, what is the paramagnetic susceptibility at $T = 0$? Give your answer in terms of the mass m of the fermions, the gyromagnetic ratio γ, the number density σ, and whatever fundamental constants (e.g., $\hbar$, c, ...) are necessary to completely specify the system.

(b) Now, suppose that the external magnetic field is replaced by an effective field H_{eff}, where

$$H_{\text{eff}} = H + \Gamma \frac{M}{A}$$

where H is the external field and M/A is the induced magnetic moment per unit area. Above what threshold value of the parameter Γ is this two-dimensional system ferromagnetic at $T = 0$?

a. We want to find χ, which is defined by $M_o = \chi H$ where $M_o = M/A$ and M is the total magnetization.

$$\epsilon = \frac{p^2}{2m} = \frac{\hbar^2}{2m}k^2 = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2 = \frac{\hbar^2\pi^2}{2mL^2}n^2$$

$$\Rightarrow n^2 = \frac{2mL^2}{\hbar^2\pi^2}\epsilon \quad \Rightarrow n = \sqrt{2m}\frac{L}{\hbar\pi}\epsilon^{1/2}$$

$$\Rightarrow dn = \frac{1}{2}\sqrt{2m}\frac{L}{\hbar\pi}\epsilon^{-1/2}d\epsilon$$

In 2D, $\rho(\epsilon)d\epsilon = \underbrace{2}_{\text{for spin}}\frac{1}{4}2\pi n\,dn = \cancel{2}\frac{\pi}{\cancel{4}}\frac{1}{\cancel{2}}2m\left(\frac{L}{\hbar\pi}\right)^2 d\epsilon = \frac{mA}{\pi\hbar^2}d\epsilon$

$$\Rightarrow \rho(\epsilon) = \frac{mA}{\pi\hbar^2} \quad \text{so } \rho(\epsilon) \text{ is constant.}$$

$\bar{M} = \sum_r \mu_{H_r}\bar{n}_r = \sum_{\pm}\mu_{H\pm}\sum_p n_p$ where p runs over momentum states

$= \mu_H\sum_p \bar{n}_p^{(+)} - \mu_H\sum_p \bar{n}_p^{(-)} \cong \mu_H\int_0^\infty \bar{n}(\epsilon_+)\rho(\epsilon_+)d\epsilon - \mu_H\int_0^\infty \bar{n}(\epsilon_-)\rho(\epsilon_-)d\epsilon$

$= \mu_H\rho(0)\int_0^\infty \bar{n}(\epsilon_+)d\epsilon - \mu_H\rho(0)\int_0^\infty \bar{n}(\epsilon_-)d\epsilon$ since ρ is constant

$= \mu_H\rho(0)\int_0^\infty \bar{n}(\epsilon - \mu_H H)d\epsilon - \mu_H\rho(0)\int_0^\infty \bar{n}(\epsilon + \mu_H H)d\epsilon$ since $U = -\vec{\mu}_H\cdot\vec{H}$

$= \mu_H\rho(0)\int_0^\infty \theta(\mu - (\epsilon - \mu_H H))d\epsilon - \mu_H\rho(0)\int_0^\infty \theta(\mu - (\epsilon + \mu_H H))d\epsilon$ at $T=0$

$= \mu_H\rho(0)(\mu + \mu_H H) - \mu_H\rho(0)(\mu - \mu_H H)$

$= \mu_H\rho(0)(2\mu_H H) = 2\mu_H^2 H\rho(0) = 2\mu_H^2 H\frac{mA}{\pi\hbar^2}$

$$\chi = \frac{M_o}{H} = \frac{M}{AH} = \frac{2m\mu_H^2}{\pi\hbar^2}$$

b. A ferromagnetic substance has $\chi \gg 1$.

$$M = 2\mu_H^2\left(H + \Gamma\frac{M}{A}\right)\frac{mA}{\pi\hbar^2} \Rightarrow \frac{1}{2}\frac{\pi\hbar^2}{mA\mu_H^2}M = H + \Gamma\frac{M}{A}$$

$$\Rightarrow H = \left(\frac{1}{2}\frac{\pi\hbar^2}{m\mu_H^2} - \Gamma\right)\frac{M}{A} \Rightarrow \chi = \frac{M}{AH} = \left(\frac{1}{2}\frac{\pi\hbar^2}{m\mu_H^2} - \Gamma\right)^{-1}$$

$\Rightarrow \Gamma \cong \frac{1}{2}\frac{\pi\hbar^2}{m\mu_H^2}$ will produce ferromagnetism