2. *Quantum Mechanics* (Spring 2006)

The Hamiltonian for a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Let $|\psi_n\rangle$, $n = 0, 1, 2, \ldots$, be the usual energy eigenstates.

(a) Suppose the system is in a state $|\phi\rangle$ that is some linear combination of the two lowest states only:

$$|\phi\rangle = c_0 |\psi_0\rangle + c_1 |\psi_1\rangle$$

and suppose it is known that the expectation value of the energy is $\hbar\omega$. What are $|c_0|$ and $|c_1|$?

(b) Choose c_0 to be real and positive, but let c_1 have any phase: $c_1 = |c_1|e^{i\theta_1}$. Suppose further that not only is the expectation value of H known to be $\hbar\omega$, but the expectation value of x is also known:

$$\langle\phi|x|\phi\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$$

What is θ_1?

(c) Now suppose the system is in the state $|\phi\rangle$ described above at time $t = 0$. That is, $|\psi(0)\rangle = |\phi\rangle$. What is $|\psi(t)\rangle$ at a later time t? Calculate the expectation value of x as a function of t. With what angular frequency does it oscillate?

a. For the simple harmonic oscillator, $H|\Psi_n\rangle = (n+\frac{1}{2})\hbar\omega$

$$\hbar\omega = \langle\Phi|H|\Phi\rangle = \left(\langle\Psi_0|c_0^* + \langle\Psi_1|c_1^*\right)H\left(c_0|\Psi_0\rangle + c_1|\Psi_1\rangle\right)$$

$$= |c_0|^2\langle\Psi_0|H|\Psi_0\rangle + |c_1|^2\langle\Psi_1|H|\Psi_1\rangle \quad \text{by orthogonality}$$

$$= |c_0|^2\left(\tfrac{1}{2}\hbar\omega\right) + |c_1|^2\left(\tfrac{3}{2}\hbar\omega\right)$$

$$\Rightarrow 1 = \tfrac{1}{2}|c_0|^2 + \tfrac{3}{2}|c_1|^2$$

Normalization of $|\Phi\rangle$ implies $\langle\Phi|\Phi\rangle = 1 \Rightarrow |c_0|^2 + |c_1|^2 = 1$

$$\Rightarrow 1 = \tfrac{1}{2}|c_0|^2 + \tfrac{3}{2}(1 - |c_0|^2) = \tfrac{3}{2} - |c_0|^2$$

$$\Rightarrow |c_0|^2 = \tfrac{1}{2} \quad \text{and} \quad |c_1|^2 = \tfrac{1}{2}$$

Therefore $|c_0| = \frac{1}{\sqrt{2}}$ and $|c_1| = \frac{1}{\sqrt{2}}$

b. Recall $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^+)$

$$\frac{1}{2}\sqrt{\frac{\hbar}{m\omega}} = \langle\Phi|x|\Phi\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle\Psi_0|c_0^* c_1 a|\Psi_1\rangle + \langle\Psi_1|c_1^* c_0 a^+|\Psi_0\rangle\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[c_0^* c_1 + c_1^* c_0\right]$$

$$= \frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{2m\omega}}\left[c_1 + c_1^*\right] \quad \text{Since } c_0 \text{ real and pos} \Rightarrow c_0 = \frac{1}{\sqrt{2}}$$

$$\Rightarrow 1 = c_1 + c_1^* = \frac{1}{\sqrt{2}}e^{i\theta_1} + \frac{1}{\sqrt{2}}e^{-i\theta_1} = \frac{1}{\sqrt{2}}2\cos(\theta_1)$$

$$\Rightarrow \cos(\theta_1) = \frac{\sqrt{2}}{2} \Rightarrow \theta_1 = \pi/4$$

c. $|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi(0)\rangle = \frac{1}{\sqrt{2}}e^{-i\omega t/2}|\Psi_0\rangle + \frac{1}{\sqrt{2}}e^{-3i\omega t/2 + i\pi/4}|\Psi_1\rangle$

$\langle\Psi(t)|x|\Psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\frac{1}{2}e^{-i\omega t + i\pi/4} + \frac{1}{2}e^{i\omega t - i\pi/4}\right] = \sqrt{\frac{\hbar}{2m\omega}}\cos\left(\omega t - \frac{\pi}{4}\right)$

The angular frequency of oscillation is ω.