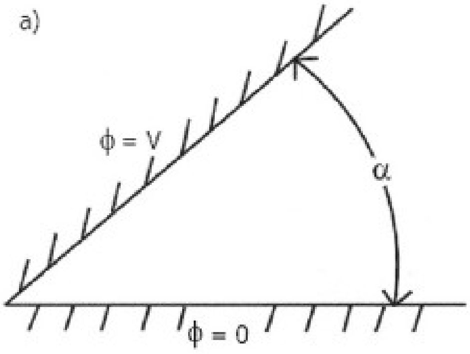
8. *Electricity and Magnetism* (Spring 2005)

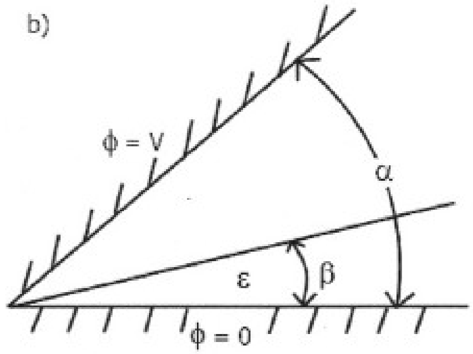
Consider a two-dimensional (r, θ) electrostatic problem consisting of two infinite plates making an angle α with each other and held at a potential difference V, as shown below:

(a) Find the potential $\phi(r, \theta)$ in the vacuum region between the plates.

Now insert a wedge dielectric, of dielectric coefficient ϵ, and angle β, resting on the bottom plate as shown below:

(b) Find the pressure experienced by the bottom plate at a distance r from the apex (from the line joining the two plates).





a. We must solve $\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} = 0$ in a cylindrical geometry

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \cancel{\frac{\partial^2 \Phi}{\partial z^2}} = 0 \qquad \frac{\partial^2 \Phi}{\partial z^2} = 0 \text{ by symmetry}$$

We seek solutions of the form $\Phi(r,\theta) = R(r)\,Q(\theta)$

$$\Rightarrow \frac{Q}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R}{r^2}\frac{\partial^2 Q}{\partial \theta^2} = 0 \qquad \text{and dividing by } \Phi,$$

$$\Rightarrow \frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{Qr^2}\frac{\partial^2 Q}{\partial \theta^2} = 0 \qquad \text{and multiplying by } r^2,$$

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{Q}\frac{\partial^2 Q}{\partial \theta^2} = 0$$

$$\Rightarrow r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \lambda^2 R \quad \text{and} \quad \frac{\partial^2 Q}{\partial \theta^2} = -\lambda^2 Q$$

$$\Rightarrow \begin{cases} \lambda \neq 0: & R(r) = Ar^{\lambda} + Br^{-\lambda} \quad \text{and} \quad Q(\theta) = C\sin(\lambda\theta) + D\cos(\lambda\theta) \\ \text{or } \lambda = 0: & R(r) = A\ln(r) + B \quad \text{and} \quad Q(\theta) = C\theta + D \end{cases}$$

B.C $\Rightarrow \lambda = 0$ and $A = 0$ and $D = 0$, then $BC = \frac{V}{\alpha} \Rightarrow \Phi(r,\theta) = \frac{V}{\alpha}\theta$

b. The pressure is caused by the force from the wedge due to its polarization charge at $\theta = 0$ and $\theta = \beta$ interacting with the field,

$$-V = \int_0^{\alpha} \vec{E}\cdot d\vec{l} = \int_0^{\beta} E_\theta \, dl + \int_{\beta}^{\alpha} E_\theta \, dl \quad \text{where } dl = r\,d\theta$$

$$-V = r\beta E_\theta^{(\epsilon)} + r(\alpha - \beta)E_\theta^{(\epsilon_0)} = r\beta\frac{D_\theta}{\epsilon} + r(\alpha-\beta)\frac{D_\theta}{\epsilon_0}$$

$$\Rightarrow D_\theta = -\frac{V}{r}\left[\beta\left(\frac{1}{\epsilon} - \frac{1}{\epsilon_0}\right) + \frac{\alpha}{\epsilon_0}\right]^{-1}$$

Let $E_\theta^0 = E_\theta(\theta < 0) = 0$, $E_\theta^1 = E_\theta(0 < \theta < \beta) = \epsilon D_\theta$, $E_\theta^2 = E_\theta(\beta < \theta < \alpha) = \epsilon_0 D_\theta$

$$P_\beta = \sigma_\beta E_{avg}(\beta) = \frac{E_\theta^2 - E_\theta^1}{\epsilon_0}\frac{1}{2}(E_\theta^2 + E_\theta^1) = \frac{1}{2\epsilon_0}\left[(E_\theta^2)^2 - (E_\theta^1)^2\right] = \frac{1}{2\epsilon_0}(\epsilon_0^2 - \epsilon^2)\frac{V^2}{r^2}\left[\beta\left(\frac{1}{\epsilon} - \frac{1}{\epsilon_0}\right) + \frac{\alpha}{\epsilon_0}\right]^{-2}$$

$$P_0 = \sigma_0 E_{avg}(0) = \frac{E_\theta^1 - E_\theta^0}{\epsilon_0}\frac{1}{2}(E_\theta^1 + E_\theta^0) = \frac{1}{2\epsilon_0}\left[(E_\theta^1)^2 - (E_\theta^0)^2\right] = \frac{1}{2\epsilon_0}\epsilon^2\frac{V^2}{r^2}\left[\beta\left(\frac{1}{\epsilon} - \frac{1}{\epsilon_0}\right) + \frac{\alpha}{\epsilon_0}\right]^{-2}$$

$$P = P_\beta + P_0 = \frac{1}{2}\epsilon_0\frac{V^2}{r^2}\left[\beta\left(\frac{1}{\epsilon} - \frac{1}{\epsilon_0}\right) + \frac{\alpha}{\epsilon_0}\right]^{-2} = \frac{1}{2}\frac{V^2}{r^2}\frac{\epsilon_0^3}{(\alpha - \beta + \beta\epsilon_0/\epsilon)^2}$$