2. *Quantum Mechanics* (Spring 2005)

Show that in one space dimension any attractive potential, no matter how weak, always has at least one bound state. *Hint:* Use the variational principle with some appropriate trial wave function such as the normalized Gaussian

$$\psi(x) = \left(\frac{2b}{\pi}\right)^{1/4} e^{-bx^2}$$

where $b$ is a parameter.

This is Shankar 5.2.2 b

We must assume as Shankar does that an attractive potential is everywhere less than its limiting values as $x \to \pm\infty$.

Then we define $V(\pm\infty) = 0$ so that $V(x) = -|V(x)|$ for all $x$.

$$H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - |V(x)|$$

$$\Rightarrow E(b) = \langle \psi_b | H | \psi_b \rangle = \langle \psi_b | -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - |V(x)| | \psi_b \rangle$$

where $\psi_b(x) = \left(\frac{2b}{\pi}\right)^{1/4} e^{-bx^2}$

$$\Rightarrow E(b) = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \left(\frac{2b}{\pi}\right)^{1/2} e^{-bx^2} \frac{\partial^2}{\partial x^2} e^{-bx^2} dx - \langle \psi_b | |V(x)| | \psi_b \rangle$$

$$= -\frac{\hbar^2}{2m}\left(\frac{2b}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-bx^2} \frac{\partial}{\partial x}\left(-2bx e^{-bx^2}\right) dx - \langle \psi_b | |V(x)| | \psi_b \rangle$$

$$= -\frac{\hbar^2}{2m}\left(\frac{2b}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-bx^2}\left(-2b e^{-bx^2} + 4b^2x^2 e^{-bx^2}\right) dx - \langle \psi_b | |V(x)| | \psi_b \rangle$$

$$= -\frac{\hbar^2}{2m}\left(\frac{2b}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} \left(4b^2x^2 - 2b\right) e^{-2bx^2} dx - \langle \psi_b | |V(x)| | \psi_b \rangle$$

$$= -\frac{\hbar^2}{2m}\cancel{\left(\frac{2b}{\pi}\right)^{1/2}}\left(4b^2 \frac{1}{4b}\cancel{\sqrt{\frac{\pi}{2b}}} - 2b\cancel{\sqrt{\frac{\pi}{2b}}}\right) - \langle \psi_b | |V(x)| | \psi_b \rangle$$

$$= \frac{\hbar^2 b}{2m} - \int_{-\infty}^{\infty} \left(\frac{2b}{\pi}\right)^{1/2} e^{-2bx^2} |V(x)| dx$$

In the limit of very small $b$,

$$E(b) \cong \frac{\hbar^2 b}{2m} - \left(\frac{2b}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} |V(x)| dx$$

We have a bound state iff $E(b) < 0$ iff

$$\frac{\hbar^2 b}{2m} - \left(\frac{2b}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} |V(x)| dx < 0$$

$$\Leftrightarrow \frac{\hbar^2 b}{2m}\left(\frac{\pi}{2b}\right)^{1/2} < \int_{-\infty}^{\infty} |V(x)| dx$$

$$\Leftrightarrow \frac{\hbar^2\sqrt{\pi}}{2\sqrt{2}m} b^{1/2} < \int_{-\infty}^{\infty} |V(x)| dx$$

$$\Leftrightarrow b < \frac{8m^2}{\pi\hbar^4}\left[\int_{-\infty}^{\infty} |V(x)| dx\right]^2$$

So whenever $b$ satisfies this condition and is very small we have a bound state. Then the variational principle tells us that the ground state has an energy less than this, so the ground state also has an energy less than zero, so the ground state must be a bound state.