14. *Statistical Mechanics and Thermodynamics* (Spring 2005)

A photon gas in thermal equilibrium is contained within a box of volume V at temperature T.

(a) Use the partition function to find the average number of photons $\bar{n}_r$ in the state having energy E_r.

(b) Find a relationship between the radiation pressure P and the energy density u (i.e. the average energy per unit volume).

(c) If the volume containing the photon gas is decreased adiabatically by a factor of 8, what is the final pressure if the initial pressure is P_o?

a. $Z = \sum_r e^{-\beta E_r} = \sum_r e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}$

$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2+\cdots)}$

$= \left(\sum_{n_1=0}^{\infty} e^{-\beta n_1\epsilon_1}\right)\left(\sum_{n_2=0}^{\infty} e^{-\beta n_2\epsilon_2}\right)\cdots$

$= \left(\frac{1}{1-e^{-\beta\epsilon_1}}\right)\left(\frac{1}{1-e^{-\beta\epsilon_2}}\right)\cdots$

$\Rightarrow \ln(Z) = \sum_{r=1}^{\infty} \ln\left(\frac{1}{1-e^{-\beta\epsilon_r}}\right) = -\sum_{r=1}^{\infty}\ln(1-e^{-\beta\epsilon_r})$

Therefore $\bar{n}_r = -\frac{1}{\beta}\frac{\partial \ln(Z)}{\partial \epsilon_r} = +\frac{1}{\cancel{\beta}}\frac{1}{1-e^{-\beta\epsilon_r}}\cancel{\beta} e^{-\beta\epsilon_r} = \frac{1}{e^{\beta\epsilon_r}-1}$

b. $P = \frac{1}{\beta}\frac{\partial \ln(Z)}{\partial V} = -\frac{1}{\cancel{\beta}}\sum_{r=1}^{\infty}\frac{1}{1-e^{-\beta\epsilon_r}}\cancel{\beta}\frac{\partial\epsilon_r}{\partial V}e^{-\beta\epsilon_r}$

$= -\sum_{r=1}^{\infty}\frac{1}{e^{\beta\epsilon_r}-1}\frac{\partial\epsilon_r}{\partial V} = -\sum_{r=1}^{\infty}\bar{n}_r\frac{\partial\epsilon_r}{\partial V}$

Now $\epsilon_r = \hbar k c = \hbar c\sqrt{\left(\frac{n_x\pi}{L}\right)^2 + \left(\frac{n_y\pi}{L}\right)^2 + \left(\frac{n_z\pi}{L}\right)^2} = \frac{\hbar c\pi}{V^{1/3}}\sqrt{n_x^2+n_y^2+n_z^2}$

$\Rightarrow \frac{\partial\epsilon_r}{\partial V} = -\frac{1}{3}\frac{\hbar c\pi}{V^{4/3}}\sqrt{n_x^2+n_y^2+n_z^2} = -\frac{\epsilon_r}{3V}$

$\Rightarrow P = \sum_{r=1}^{\infty}\bar{n}_r\frac{\epsilon_r}{3V} = \frac{1}{3V}\sum_{r=1}^{\infty}\bar{n}_r\epsilon_r = \frac{E}{3V} = \frac{1}{3}u$

c. $dE = -p\,dV$ since $đQ = 0$ for an adiabatic process

$\Rightarrow d(uV) = -\frac{u}{3}dV$

$\Rightarrow u\,dV + V\,du = -\frac{u}{3}dV$

$\Rightarrow \frac{dV}{V} + \frac{du}{u} = -\frac{1}{3}\frac{dV}{V}$

$\Rightarrow \frac{du}{u} = -\frac{4}{3}\frac{dV}{V}$

$\Rightarrow \ln(u) = -\frac{4}{3}\ln(V) + C$

$\Rightarrow u = A\,V^{-4/3}$

$\Rightarrow P = A'\,V^{-4/3}$ so this is a polytropic process

$\Rightarrow P_o V_o^{4/3} = P_f V_f^{4/3} = A' \Rightarrow P_f = P_o\left(\frac{V_o}{V_f}\right)^{4/3} = P_o(8)^{4/3} = 16P_o$