1. *Quantum Mechanics* (Spring 2005)

Consider a particle of charge q in a one-dimensional harmonic oscillator potential. Suppose there is also a weak electric field E so that the potential is shifted by

$$H' = -qEx$$

(a) Calculate the correction to the simple harmonic oscillator energy levels through second order in perturbation theory.

(b) Now solve the problem exactly. How do the exact energy levels compare with the perturbative result in (a)?

a. Recall $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^{+})$

$$\Delta_n^{(1)} = H'_{nn} = \langle \Psi_n | H' | \Psi_n \rangle = -qE \langle \Psi_n | x | \Psi_n \rangle = 0$$

$$\Delta_n^{(2)} = \sum_{k\neq n} \frac{|H'_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} = \sum_{k\neq n} \frac{|\langle \Psi_n | H' | \Psi_k \rangle|^2}{(n+\frac{1}{2})\hbar\omega - (k+\frac{1}{2})\hbar\omega}$$

$$= \frac{q^2E^2}{\hbar\omega} \sum_{k\neq n} \frac{|\langle \Psi_n | x | \Psi_k \rangle|^2}{n-k}$$

$$= \frac{q^2E^2}{\hbar\omega}\,\frac{\hbar}{2m\omega} \sum_{k\neq n} \frac{|\langle \Psi_n | a + a^{+} | \Psi_k \rangle|^2}{n-k}$$

$$= \frac{q^2E^2}{2m\omega^2} \sum_{k\neq n} \frac{|\sqrt{k}\,\delta_{n,k-1} + \sqrt{k+1}\,\delta_{n,k+1}|^2}{n-k}$$

$$= \frac{q^2E^2}{2m\omega^2}\left(\frac{n+1}{-1} + \frac{n}{1}\right) = -\frac{q^2E^2}{2m\omega^2}$$

b. Complete the square in the potential.

$$V = \frac{1}{2}m\omega^2x^2 - qEx \Rightarrow \frac{2}{m\omega^2}V = x^2 - \frac{2}{m\omega^2}qEx + \left(\frac{q^2E^2}{m^2\omega^4} - \frac{q^2E^2}{m^2\omega^4}\right)$$

$$\Rightarrow \frac{2}{m\omega^2}V = \left(x - \frac{qE}{m\omega^2}\right)^2 - \frac{q^2E^2}{m^2\omega^4}$$

$$\Rightarrow V = \frac{1}{2}m\omega^2\left(x - \frac{qE}{m\omega^2}\right)^2 - \frac{q^2E^2}{2m\omega^2}$$

So the exact shift is $\Delta_n = -\frac{q^2E^2}{2m\omega^2}$, which is exactly the perturbative result found in part a.