5. *Quantum Mechanics* (Spring 2005)

Calculate the transmission coefficient for a particle of energy $E > 0$ scattering off the $1D$ potential well $V(x) = V_0$ for $0 < x < a$, $V(x) = 0$ elsewhere, $V_0 < 0$. Are there resonance phenomena?

See Griffiths Section 2.6

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi$$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} = -\frac{2m(E-V)}{\hbar^2}\psi$$

Let $k = \sqrt{\frac{2mE}{\hbar^2}}$ and $\ell = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$ which is real because $V_0 < 0 < E$

$\Rightarrow \psi_1(x) = Ae^{ikx} + Be^{-ikx}$, $\psi_2(x) = Ce^{i\ell x} + De^{-i\ell x}$, $\psi_3(x) = Fe^{ikx} + \cancel{Ge^{-ikx}}$

$G = 0$ since there is no wave coming from the right.

Since $V(x) < \infty$ $\forall x$, we impose continuity on $\psi(x)$ and $\psi'(x)$:

$$\psi_1(0) = \psi_2(0) \Rightarrow A + B = C + D$$

$$\psi_1'(0) = \psi_2'(0) \Rightarrow k(A-B) = \ell(C-D)$$

$$\psi_2(a) = \psi_3(a) \Rightarrow Ce^{i\ell a} + De^{-i\ell a} = Fe^{ika}$$

$$\psi_2'(a) = \psi_3'(a) \Rightarrow \ell(Ce^{i\ell a} - De^{-i\ell a}) = kFe^{ika}$$

Now use the second two equations to solve for C and D:

$$2Ce^{i\ell a} = \left(1+\frac{k}{\ell}\right)Fe^{ika} \Rightarrow C = \frac{1}{2}\left(1+\frac{k}{\ell}\right)Fe^{ika}e^{-i\ell a}$$

$$2De^{-i\ell a} = \left(1-\frac{k}{\ell}\right)Fe^{ika} \Rightarrow D = \frac{1}{2}\left(1-\frac{k}{\ell}\right)Fe^{ika}e^{i\ell a}$$

Next use the first two equations to eliminate B and insert C, D:

$$2A = C + D + \frac{\ell}{k}(C-D) \Rightarrow A = \frac{1}{2}\left(1+\frac{\ell}{k}\right)C + \frac{1}{2}\left(1-\frac{\ell}{k}\right)D$$

$$\Rightarrow A = \frac{1}{4}\left(2+\frac{k}{\ell}+\frac{\ell}{k}\right)Fe^{ika}e^{-i\ell a} + \frac{1}{4}\left(2-\frac{k}{\ell}-\frac{\ell}{k}\right)Fe^{ika}e^{i\ell a}$$

$$= Fe^{ika}\cos(\ell a) - \frac{i}{2}\frac{k^2+\ell^2}{k\ell}Fe^{ika}\sin(\ell a)$$

$$\Rightarrow F = Ae^{-ika}\left[\cos(\ell a) - \frac{i}{2}\frac{k^2+\ell^2}{k\ell}\sin(\ell a)\right]^{-1}$$

Therefore $T \equiv \frac{|F|^2}{|A|^2} = \left[\cos^2(\ell a) + \frac{1}{4}\left(\frac{k^2+\ell^2}{k\ell}\right)^2\sin^2(\ell a)\right]^{-1}$

$$= \left[1 + \left(\frac{1}{4}\frac{k^4+2k^2\ell^2+\ell^4}{k^2\ell^2} - \frac{4k^2\ell^2}{4k^2\ell^2}\right)\sin^2(\ell a)\right]^{-1} \quad \left(\cos^2(\ell a) = 1-\sin^2(\ell a)\right)$$

$$= \left[1 + \frac{1}{4}\left(\frac{k^2-\ell^2}{k\ell}\right)^2\sin^2(\ell a)\right]^{-1}$$

$$= \left[1 + \frac{1}{4}\left(\frac{V_0}{\sqrt{E(E-V_0)}}\right)^2\sin^2\left(\frac{a}{\hbar}\sqrt{2m(E-V_0)}\right)\right]^{-1}$$

$$= \left[1 + \frac{1}{4}\frac{V_0^2}{E(E-V_0)}\sin^2\left(\frac{a}{\hbar}\sqrt{2m(E-V_0)}\right)\right]^{-1}$$

Resonance phenomena can occur if the energy is just right.

The Ramsauer-Townsend effect gives perfect transmission

so $T = 1 \Leftrightarrow \frac{a}{\hbar}\sqrt{2m(E-V_0)} = n\pi \Leftrightarrow 2m(E-V_0) = \left(\frac{n\pi\hbar}{a}\right)^2 \Leftrightarrow E = \frac{n^2\pi^2\hbar^2}{2ma^2} + V_0$