4. *Quantum Mechanics* (Spring 2004)

The electron neutrino $|\nu_e\rangle$ and the muon neutrino $|\nu_\mu\rangle$ are the possible neutrino states produced and detected in experiments, but they are not necessarily eigenstates of the Hamiltonian. Rather, if the state is known to have momentum p, it is some linear combination of the energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ of the form

$$|\nu_e\rangle = \cos(\theta)\,|\nu_1\rangle + \sin(\theta)\,|\nu_2\rangle$$
$$|\nu_\mu\rangle = -\sin(\theta)\,|\nu_1\rangle + \cos(\theta)\,|\nu_2\rangle$$

where

$$H\,|\nu_1\rangle = \sqrt{p^2c^2 + m_1^2c^4}\,|\nu_1\rangle$$
$$H\,|\nu_2\rangle = \sqrt{p^2c^2 + m_2^2c^4}\,|\nu_2\rangle$$

for two possibly different masses m_1 and m_2, and some "mixing angle" θ. If it is known that a neutrino was definitely a ν_μ when it was produced, what is the probability of detecting a ν_e after it has traveled a distance L? Assume that $m_1c \ll p$ and $m_2c \ll p$, so that the neutrinos are moving at almost (or even exactly) the speed of light, (so you can ignore corrections of the order $1 - v/c$ compared to terms of order 1) and state your result to first order in the difference $\Delta m^2 = m_1^2 - m_2^2$.

This is a simplified version of an actual neutrino oscillation experiment like the super-Kamiokande detector experiment a few years ago. In reality there is a third neutrino $|\nu_\tau\rangle$.

$$|\Psi(t)\rangle = U(t)\,|\Psi(0)\rangle = e^{-iHt/\hbar}\,|\Psi(0)\rangle = e^{-iHt/\hbar}\,|\nu_\mu\rangle$$
$$= -\sin(\theta)\, e^{-iHt/\hbar}\,|\nu_1\rangle + \cos(\theta)\, e^{-iHt/\hbar}\,|\nu_2\rangle$$
$$= -\sin(\theta)\exp\left[-i\sqrt{p^2c^2+m_1^2c^4}\,t/\hbar\right]|\nu_1\rangle + \cos(\theta)\exp\left[-i\sqrt{p^2c^2+m_2^2c^4}\,t/\hbar\right]|\nu_2\rangle$$

$$\langle \nu_e|\Psi(t)\rangle = -\sin(\theta)\cos(\theta)\exp\left[-i\sqrt{p^2c^2+m_1^2c^4}\,t/\hbar\right] + \sin(\theta)\cos(\theta)\exp\left[-i\sqrt{p^2c^2+m_2^2c^4}\,t/\hbar\right]$$

Now $\sqrt{1+x} \cong 1 + \frac{x}{2}$ for small x, so $\sqrt{p^2c^2+m^2c^4} = pc\sqrt{1+\frac{m^2c^2}{p^2}} \cong pc\left(1+\frac{m^2c^2}{2p}\right)$

since we are assuming $m_1c \ll p$ and $m_2 c \ll p$

$$\langle \nu_e|\Psi(t)\rangle \cong \tfrac{1}{2}\sin(2\theta)\left\{\exp\left[-ipc\left(1+\frac{m_2^2c^2}{2p^2}\right)t/\hbar\right] - \exp\left[-ipc\left(1+\frac{m_1^2c^2}{2p^2}\right)t/\hbar\right]\right\}$$

$$P(\nu_e) = |\langle \nu_e|\Psi(t)\rangle|^2 \cong \tfrac{1}{4}\sin^2(2\theta)\left\{1 - \exp\left[i\,\frac{m_2^2-m_1^2}{2p}c^3t/\hbar\right] - \exp\left[i\,\frac{m_1^2-m_2^2}{2p}c^3t/\hbar\right] + 1\right\}$$
$$= \tfrac{1}{4}\sin^2(2\theta)\left\{2 - 2\cos\left(\tfrac{1}{2}\,\frac{\Delta m^2}{p}c^3t/\hbar\right)\right\}$$
$$= \tfrac{1}{2}\sin^2(2\theta)\left\{1 - \cos\left(\tfrac{1}{2}\,\frac{\Delta m^2c^2}{p}L/\hbar\right)\right\} \quad \text{since } t = \frac{L}{c}$$

Now $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta)$

$\Rightarrow 1 - \cos(2\theta) = 2\sin^2(\theta) \Rightarrow 1 - \cos(\theta) = 2\sin^2(\theta/2)$

$$P(\nu_e) = \sin^2(2\theta)\,\sin^2\left(\frac{\Delta m^2c^2}{4p}L/\hbar\right)$$