2. *Quantum Mechanics* (Spring 2004)

A hydrogen atom is in the ground state ($n = 1, l = m = 0$) for $t < 0$. Suppose the atom is placed between the plates of a capacitor, and a weak, spatially uniform but time-dependent decaying field is applied at $t = 0$. The field (for $t > 0$) is

$$\mathbf{E} = \mathbf{E}_o e^{-\gamma t}$$

for some $\gamma > 0$. Take $\mathbf{E}_o$ along the z-axis. What is the probability (to first order in E_o) that the atom will be in each of the four $n = 2$ states as $t \to \infty$? Neglect spin.

You may need some of the functions $R_{nl}(r)$ and $Y_l^m(\theta, \phi)$ in the following table:

$$a^{\frac{3}{2}} R_{10}(r) = 2e^{-r/a} \quad a^{\frac{3}{2}} R_{20}(r) = \frac{1}{\sqrt{2}}\left(1 - \frac{r}{2a}\right) e^{-r/2a} \quad a^{\frac{3}{2}} R_{21}(r) = \frac{1}{2\sqrt{6}} \frac{r}{a} e^{-r/2a}$$

$$Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \quad Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos(\theta) \quad Y_1^{\pm 1}(\theta, \phi) = \mp\sqrt{\frac{3}{8\pi}} \sin(\theta) e^{\pm i\phi}$$

Table 1: Some hydrogen atom radial wave functions and spherical harmonics. a is the Bohr radius: $a = \hbar/mc\alpha$.

This is Abers problem 9.1

And an integral

$$\int_0^\infty x^n e^{-x/a}\, dx = a^{n+1} n!$$

$P(f) = |\langle \Phi_f | U(t) | \Phi_i \rangle|^2 = |\langle \Psi_f | \Psi \rangle|^2$ where $|\Psi\rangle = U(t)|\Phi_i\rangle$ and $|\Psi_f\rangle = e^{-iH^\circ t/\hbar}|\Phi_f\rangle$. H° is the unperturbed hydrogen atom Hamiltonian and $H = H^\circ + H'$ where $H' = -eEz = -eE_o e^{-\gamma t} z$

To derive the formula we need, start with the time-dependent S.E.

$$H(t)|\Psi\rangle = i\hbar|\dot{\Psi}\rangle \Rightarrow \langle \Psi_f | H(t) | \Psi \rangle = i\hbar \langle \Psi_f | \dot{\Psi} \rangle$$

$$\Rightarrow \langle \Psi_f | H(t) | \Psi \rangle = i\hbar \left[\frac{\partial}{\partial t}\langle \Psi_f | \Psi \rangle - \langle \dot{\Psi}_f | \Psi \rangle\right]$$

$$\Rightarrow \langle \Psi_f | H^\circ | \Psi \rangle + \langle \Psi_f | H'(t) | \Psi \rangle = i\hbar \frac{\partial}{\partial t}\langle \Psi_f | \Psi \rangle - i\hbar\left(\frac{i}{\hbar}\langle \Psi_f | H^\circ\right)|\Psi\rangle$$

$$\Rightarrow \langle \Psi_f | H'(t) | \Psi \rangle = i\hbar \frac{\partial}{\partial t}\langle \Psi_f | \Psi \rangle \Rightarrow \langle \Psi_f | \Psi \rangle = \langle \Psi_f | \Psi \rangle|_{t=0} - \frac{i}{\hbar}\int_0^t \langle \Psi_f | H'(t') | \Psi \rangle dt'$$

Approximate $|\Psi\rangle$ in the integrand as $|\Psi_i\rangle$ like the Born approximation.

$$\Rightarrow \langle \Psi_f | \Psi \rangle \cong \delta_{fi} - \frac{i}{\hbar}\int_0^t \langle \Phi_f | H'(t') | \Phi_i \rangle e^{i\omega_{fi} t'} dt'$$

For our problem, the perturbation is the 0th spherical component of a rank 1 tensor so $|1 - \ell| \le \ell' \le 1 + \ell \Rightarrow \ell' = 1$ and $m' = m + 0 \Rightarrow m' = 0$ by the Wigner-Eckart Theorem selection rules, so only $|\Phi_f\rangle = |210\rangle$ is nonzero.

$$\langle 210 | z | 100 \rangle = \int_0^{2\pi}\int_0^\pi\int_0^\infty R_{21}(r) Y_1^\circ(\theta,\phi)\, r\cos(\theta)\, R_{10}(r)\, Y_0^\circ(\theta,\phi)\, r^2 \sin(\theta)\, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^\pi\int_0^\infty \frac{1}{\cancel{2}\sqrt{6}} a^{-5/2}\, r e^{-r/2a} \sqrt{\frac{3}{4\pi}}\cos(\theta)\, r\cos(\theta)\, \cancel{2}\, a^{-3/2} e^{-r/a} \frac{1}{\sqrt{4\pi}} r^2 \sin(\theta)\, dr\, d\theta\, d\phi$$

$$= \frac{2\pi}{4\pi}\frac{1}{\sqrt{2}} a^{-4} \int_0^\pi \cos^2(\theta)\sin(\theta)\, d\theta \int_0^\infty r^4 e^{-3r/2a} dr$$

$$= \frac{1}{2\sqrt{2}} a^{-4} \left(\frac{2}{3}\right)\left(\frac{2a}{3}\right)^5 4! = \frac{2^8 a}{3^5\sqrt{2}}$$

$$P(|210\rangle)|_{t\to\infty} = \left|\frac{-i}{\hbar}\int_0^t \langle 210 | -eE_o e^{-\gamma t'} z | 100 \rangle e^{i\omega_{fi} t'} dt'\right|^2_{t\to\infty}$$

$$= \frac{e^2 E_o^2}{\hbar^2}\left(\frac{2^{15} a^2}{3^{10}}\right)\left|\int_0^\infty e^{(i\omega_{fi} - \gamma)t'} dt'\right|^2$$

$$= \frac{2^{15}}{3^{10}}\frac{e^2 a^2 E_o^2}{\hbar^2}\left|\frac{1}{i\omega_{fi} - \gamma}(-1)\right|^2 = \frac{2^{15}}{3^{10}}\frac{e^2 a^2 E_o^2}{\hbar^2}\frac{1}{\gamma^2 + \omega_{fi}^2} \quad \left(\omega_{fi} = \frac{E_2 - E_1}{\hbar} = -\frac{3E_1}{4\hbar}\right)$$