8. *Electricity and Magnetism* (Spring 2003)

A cylindrical capacitor of length $L$ is composed of an inner cylindrical conductor of radius $r$ and a concentric outer conducting cylindrical shell of radius $R$.

(a) What is the capacitance of this arrangement (you may ignore fringing fields at the ends)?

(b) The two conductors are held at a constant potential difference, $V$, using a battery. A cylindrical shell of dielectric material of length $L$ and which just fits in between the conductors (inner radius $\sim r$ and outer radius $\sim R$) is inserted so that half is inside of the capacitor (i.e. $L/2$ of the length of the capacitor is now filled with dielectric). What is the force on the dielectric in this position (magnitude and direction)?

a. $Q=CV$, so we calculate the voltage for a given total charge. Note that a charge $Q$ on the capacitor means each plate has magnitude of charge $Q$.

We place a Gaussian cylinder around the inner cylinder with radius $\rho$. $\int_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow 2\pi\rho\,\Delta z\,E = \frac{\frac{Q}{L}\Delta z}{\epsilon_0}$

$$\Rightarrow \vec{E} = \frac{Q}{2\pi\epsilon_0\rho L}\hat{r}$$

$$V = -\int_r^R \vec{E}(\vec{r})\cdot d\vec{l} = -\int_r^R \frac{Q}{2\pi\epsilon_0 L}\frac{1}{\rho}\,d\rho = -\frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{R}{r}\right)$$

$$Q=CV \Rightarrow C = \frac{Q}{V} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{R}{r}\right)}$$ since we use the magnitude of $V$

b. We find the stored energy as a function of $z$ from $U=\frac{1}{2}CV^2$ and then find $\vec{F}$ by $\vec{F}=-\vec{\nabla}U = -\frac{\partial U}{\partial z}\hat{z}$.

If the material has dielectric constant $\epsilon$, then the capacitance for the filled part follows the same derivation as in part (a), but $\epsilon_0$ is replaced with $\epsilon$. Assume the dielectric comes in from below.

$$U(z) = \frac{1}{2}C'(z)V^2 + \frac{1}{2}C(L-z)V^2$$

$$= \frac{1}{2}\frac{2\pi\epsilon z}{\ln\left(\frac{R}{r}\right)}V^2 + \frac{1}{2}\frac{2\pi\epsilon_0(L-z)}{\ln\left(\frac{R}{r}\right)}V^2$$

$$= \frac{\pi V^2}{\ln\left(\frac{R}{r}\right)}\left(\epsilon z + \epsilon_0(L-z)\right)$$

$$\Rightarrow \vec{F} = -\frac{\partial U}{\partial z}\hat{z} = -\frac{\pi V^2}{\ln\left(\frac{R}{r}\right)}(\epsilon-\epsilon_0)\hat{z}$$

And $\epsilon = (1+\chi_e)\epsilon_0 \Rightarrow \vec{F} = -\frac{\pi\chi_e\epsilon_0}{\ln\left(\frac{R}{r}\right)}V^2\hat{z}$

which is pushing the dielectric back out.