3. *Quantum Mechanics* (Spring 2003)

A *coherent state* of a simple harmonic oscillator is an eigenstate of the annihilation operator, a. In terms of the energy eigenvalue basis, give an explicit expression for a coherent state $|\alpha\rangle$ satisfying $a|\alpha\rangle = \alpha|\alpha\rangle$.

See Griffiths Problem 3.35

First we write $|\alpha\rangle$ as an eigenvalue expansion

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle .$$

$$a|\alpha\rangle = \sum_{n=0}^{\infty} c_n a|n\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}\,|n-1\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n}\,|n-1\rangle$$ since it's zero for $n=0$

Now let $n \to n+1$ $\Rightarrow$ $a|\alpha\rangle = \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\,|n\rangle$

So $a|\alpha\rangle = \alpha|\alpha\rangle \Rightarrow \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\,|n\rangle = \alpha\sum_{n=0}^{\infty} c_n|n\rangle = \sum_{n=0}^{\infty} \alpha c_n |n\rangle$

Now take the projection with $\langle m|$, $\sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\,\langle m|n\rangle = \sum_{n=0}^{\infty} \alpha c_n \langle m|n\rangle$

$$\Rightarrow \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\,\delta_{mn} = \sum_{n=0}^{\infty} \alpha c_n \delta_{mn} \Rightarrow c_{m+1}\sqrt{m+1} = \alpha c_m$$

$$\Rightarrow c_{m+1} = \frac{\alpha c_m}{\sqrt{m+1}} \quad \text{So} \quad c_1 = \frac{\alpha c_0}{\sqrt{1}},\ c_2 = \frac{\alpha^2 c_0}{\sqrt{2}},\ c_3 = \frac{\alpha^3 c_0}{\sqrt{6}}$$

$$\Rightarrow c_m = \frac{\alpha^m c_0}{\sqrt{m!}} \quad \Rightarrow \quad |\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n c_0}{\sqrt{n!}}|n\rangle$$

Now we can determine c_0 by normalizing.

$$1 = \langle\alpha|\alpha\rangle = \left(\sum_{n=0}^{\infty} \frac{(\alpha^*)^n c_0^*}{\sqrt{n!}}\langle n|\right)\left(\sum_{n=0}^{\infty} \frac{\alpha^n c_0}{\sqrt{n!}}|n\rangle\right)$$

$$= |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2} \Rightarrow c_0 = e^{-|\alpha|^2/2}$$

Therefore $$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$