1. *Quantum Mechanics* (Spring 2003)

In one-dimension, a particle is subject to a harmonic oscillator potential with a time dependent origin,

$$V(x) = \frac{1}{2}m\omega^2(x - \epsilon(t))^2$$

where

$$\epsilon(t) = \epsilon e^{-t^2/\tau^2}, \quad \epsilon \ll 1$$

Suppose the particle is in the ground state at $t = -\infty$. What states can the particle be in at $t = +\infty$, and what are the probabilities for each? Work to lowest order in ϵ.

$$V(x) = \frac{1}{2}m\omega^2\left(x^2 - 2x\,\epsilon(t) + \cancel{\epsilon^2(t)}\right)$$

$$\Rightarrow H'(t) = -m\omega^2 x\,\epsilon(t) = -m\omega^2 x\,\epsilon\, e^{-t^2/\tau^2}$$

$$\langle\psi_f|\psi\rangle = \delta_{fi} - \frac{i}{\hbar}\int_{-\infty}^{\infty}\langle\phi_f|H'(t')|\phi_i\rangle e^{-i\omega_{fi}t'}dt'$$

$$= \delta_{n'0} + \frac{i}{\hbar}m\omega^2\epsilon\int_{-\infty}^{\infty}\langle n'|x|0\rangle e^{-in'\omega t'}e^{-t'^2/\tau^2}dt'$$

$$= \delta_{n'0} + \frac{i}{\hbar}m\omega^2\epsilon\int_{-\infty}^{\infty}\langle n'|\sqrt{\frac{\hbar}{2m\omega}}(a^+ + a)|0\rangle e^{-in'\omega t'}e^{-t'^2/\tau^2}dt'$$

$$\Rightarrow \langle\psi_1|\psi\rangle = \frac{i}{\hbar}m\omega^2\epsilon\sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{\infty}e^{-t'^2/\tau^2 - i\omega t'}dt'$$

$$\int_{-\infty}^{\infty}e^{-t'^2/\tau^2 - i\omega t'}dt' = \int_{-\infty}^{\infty}e^{-\frac{1}{\tau^2}(t'^2 + i\omega\tau^2 t' - \frac{1}{4}\omega^2\tau^4) - \frac{1}{4}\omega^2\tau^2}dt'$$

$$= \int_{-\infty}^{\infty}e^{-\frac{1}{\tau^2}(t' - \frac{i}{2}\omega\tau^2)^2 - \frac{1}{4}\omega^2\tau^2}dt' = e^{-\frac{1}{4}\omega^2\tau^2}\int_{-\infty}^{\infty}e^{-u^2/\tau^2}du$$

$$= e^{-\frac{1}{4}\omega^2\tau^2}\tau\int_{-\infty}^{\infty}e^{-y^2}dy = \tau e^{-\frac{1}{4}\omega^2\tau^2}\sqrt{\pi}$$

$$\Rightarrow \langle\psi_1|\psi\rangle = \frac{i}{\hbar}m\omega^2\epsilon\sqrt{\frac{\hbar}{2m\omega}}\,\tau e^{-\frac{1}{4}\omega^2\tau^2}\sqrt{\pi}$$

$$\Rightarrow P(n=1) = |\langle\psi_1|\psi\rangle|^2 = \frac{m^2\omega^4}{\hbar^2}\epsilon^2\frac{\hbar}{2m\omega}\tau^2 e^{-\frac{1}{2}\omega^2\tau^2}\pi$$

$$= \frac{\pi m\omega^3\tau^2\epsilon^2}{2\hbar}e^{-\frac{1}{2}\omega^2\tau^2} \quad \text{to first order}$$

and $P(n=0) = 1$ to first order

and $P(n) = 0$ to first order for all $n = 0, 1$.