1. *Quantum Mechanics* (Fall 2004)

Two spin-half particles are in a state with total spin zero. Let $\hat{\mathbf{n}}_a$ and $\hat{\mathbf{n}}_b$ be unit vectors in two arbitrary directions. Calculate the expectation value of the *product* of the spin of the first particle along $\hat{\mathbf{n}}_a$ and the spin of the second along $\hat{\mathbf{n}}_b$. That is, if $\mathbf{s}_a$ and $\mathbf{s}_b$ are the two spin operators, calculate

$$\langle\psi|\mathbf{s}_a\cdot\hat{\mathbf{n}}_a\,\mathbf{s}_b\cdot\hat{\mathbf{n}}_b|\psi\rangle$$

Hint: Because the state is spherically symmetric the answer can depend only on the angle between the two directions.

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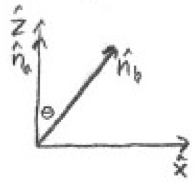
Let the z-axis lie in the direction of $\hat{n}_a$ and the x-axis in the direction of $\hat{n}_b$. Then



$$\vec{S}_a\cdot\hat{n}_a = S_{az} \quad\text{and}\quad \vec{S}_b\cdot\hat{n}_b = S_{bx}n_{bx} + S_{bz}\cos(\theta)$$

where θ is the angle between $\hat{n}_a$ and $\hat{n}_b$.

The spin-singlet state is $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{\hbar}{2},-\tfrac{\hbar}{2}\rangle - |-\tfrac{\hbar}{2},\tfrac{\hbar}{2}\rangle\right)$

$$\begin{aligned}\langle\Psi|\vec{S}_a\cdot\hat{n}_a\,\vec{S}_b\cdot\hat{n}_b|\Psi\rangle &= \langle\Psi|S_{az}\left(S_{bx}n_{bx} + S_{bz}\cos(\theta)\right)|\Psi\rangle\\ &= \langle\Psi|S_{az}S_{bx}|\Psi\rangle n_{bx} + \langle\Psi|S_{az}S_{bz}|\Psi\rangle\cos(\theta)\end{aligned}$$

$$\begin{aligned}\langle\Psi|S_{az}S_{bx}|\Psi\rangle &= \frac{1}{2}\left[\langle\tfrac{\hbar}{2},-\tfrac{\hbar}{2}| - \langle-\tfrac{\hbar}{2},\tfrac{\hbar}{2}|\right]S_{az}S_{bx}\left[|\tfrac{\hbar}{2},-\tfrac{\hbar}{2}\rangle - |-\tfrac{\hbar}{2},\tfrac{\hbar}{2}\rangle\right]\\ &= \frac{1}{2}\frac{\hbar}{2}\left[\langle\tfrac{\hbar}{2},-\tfrac{\hbar}{2}| + \langle-\tfrac{\hbar}{2},\tfrac{\hbar}{2}|\right]S_{bx}\left[|\tfrac{\hbar}{2},-\tfrac{\hbar}{2}\rangle - |-\tfrac{\hbar}{2},\tfrac{\hbar}{2}\rangle\right]\\ &= \frac{\hbar}{4}\left[(0\ 1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - (0\ 1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + (1\ 0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - (1\ 0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]\\ &= \frac{\hbar}{4}\left[0 - 1 + 1 - 0\right] = 0\end{aligned}$$

$$\begin{aligned}\langle\Psi|S_{az}S_{bz}|\Psi\rangle &= \frac{1}{2}\left[\langle\tfrac{\hbar}{2},-\tfrac{\hbar}{2}| - \langle-\tfrac{\hbar}{2},\tfrac{\hbar}{2}|\right]S_{az}S_{bz}\left[|\tfrac{\hbar}{2},-\tfrac{\hbar}{2}\rangle - |-\tfrac{\hbar}{2},\tfrac{\hbar}{2}\rangle\right]\\ &= \frac{1}{2}\frac{\hbar}{2}\left[\langle\tfrac{\hbar}{2},-\tfrac{\hbar}{2}| - \langle-\tfrac{\hbar}{2},\tfrac{\hbar}{2}|\right]S_{az}\left[-|\tfrac{\hbar}{2},-\tfrac{\hbar}{2}\rangle - |-\tfrac{\hbar}{2},\tfrac{\hbar}{2}\rangle\right]\\ &= \frac{\hbar^2}{8}\left[\langle\tfrac{\hbar}{2},-\tfrac{\hbar}{2}| - \langle-\tfrac{\hbar}{2},\tfrac{\hbar}{2}|\right]\left[-|\tfrac{\hbar}{2},-\tfrac{\hbar}{2}\rangle + |-\tfrac{\hbar}{2},\tfrac{\hbar}{2}\rangle\right]\\ &= \frac{\hbar^2}{8}\left[-\langle\tfrac{\hbar}{2},-\tfrac{\hbar}{2}|\tfrac{\hbar}{2},-\tfrac{\hbar}{2}\rangle - \langle-\tfrac{\hbar}{2},\tfrac{\hbar}{2}|-\tfrac{\hbar}{2},\tfrac{\hbar}{2}\rangle\right] = -\frac{\hbar^2}{4}\end{aligned}$$

Therefore $\langle\Psi|\vec{S}_a\cdot\hat{n}_a\,\vec{S}_b\cdot\hat{n}_b|\Psi\rangle = -\frac{\hbar^2}{4}\cos(\theta)$