6. *Statistical Mechanics and Thermodynamics* (Fall 2003)

Consider a gas of non-conserved Bosons in three dimensions. The energy-verses-momentum relationship for each of these exotic particles is $E = Ap^2$.

(a) Calculate the grand partition function of this gas. Assume a spin degeneracy factor of 1.

(b) Find the energy of this gas as a function of temperature. The energy goes as a simple power law in the temperature. What is the power?

(c) Find the pressure as a function of temperature. What power law describes the temperature dependence of the pressure?

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a. $\mathcal{Z} \equiv \sum_{N'} Z(N')\, e^{-\alpha N'} \cong Z(N)\, e^{-\alpha N} \Delta^* N'$ since narrowly peaked

$\Rightarrow \ln(\mathcal{Z}) = \ln(Z(N)) - \alpha N$ ($\Delta^* N'$ Is not important if we take log)

$\Rightarrow \mathcal{Z} = Z(N)\, e^{-\alpha N} = \left(\sum_R e^{-\beta E_R}\right) e^{-\alpha N}$

$= \sum_R e^{-\beta(\epsilon_1 n_1 + \epsilon_2 n_2 + \ldots)}\, e^{-\alpha(n_1 + n_2 + \ldots)}$

$= \sum_{n_1, n_2, \ldots} e^{-(\alpha + \beta\epsilon_1) n_1 - (\alpha + \beta\epsilon_2) - \ldots}$

$= \left(\sum_{n_1} e^{-(\alpha+\beta\epsilon_1) n_1}\right)\left(\sum_{n_2} e^{-(\alpha+\beta\epsilon_1) n_2}\right) \ldots$

$= \left(\frac{1}{1 - e^{-(\alpha+\beta\epsilon_1)}}\right)\left(\frac{1}{1 - e^{-(\alpha+\beta\epsilon_2)}}\right) \ldots$

$\Rightarrow \ln(\mathcal{Z}) = -\sum_r \ln\left(1 - e^{-(\alpha+\beta\epsilon_r)}\right)$

$= -\sum_r \ln\left(1 - e^{-\beta\epsilon_r}\right)$ since $\alpha = 0$ for non-conserved particles

$\Rightarrow \ln(\mathcal{Z}) = -\int_0^\infty \ln(1 - e^{-\beta\epsilon})\, \rho(\epsilon)\, d\epsilon$

$\rho(\epsilon) d\epsilon = \rho(\vec{n}) d^3 n = \frac{1}{8} 4\pi n^2 dn \qquad \epsilon = Ap^2 = A\hbar^2 \frac{\pi^2 n^2}{L^2}$

$\Rightarrow n^2 = \frac{L^2}{A\hbar^2\pi^2}\epsilon \Rightarrow n = \frac{L}{\hbar\pi\sqrt{A}}\epsilon^{1/2} \Rightarrow dn = \frac{L}{\hbar\pi\sqrt{A}} \frac{\epsilon^{-1/2}}{2} d\epsilon$

$\Rightarrow \rho(\epsilon) = \frac{\pi}{2} n^2 dn = \frac{\pi}{4}\left(\frac{L}{\hbar\pi\sqrt{A}}\right)^3 \epsilon^{1/2} d\epsilon$

$\Rightarrow \ln(\mathcal{Z}) = -\frac{\pi}{4}\left(\frac{L}{\hbar\pi\sqrt{A}}\right)^3 \int_0^\infty \epsilon^{1/2} \ln(1 - e^{-\beta\epsilon})\, d\epsilon$

b. Note that $\alpha = 0 \Rightarrow \ln(\mathcal{Z}) = \ln(Z)$ so

$E = -\frac{\partial \ln(Z)}{\partial\beta} = \frac{\pi}{4}\left(\frac{L}{\hbar\pi\sqrt{A}}\right)^3 \int_0^\infty \frac{\epsilon^{1/2}}{1 - e^{-\beta\epsilon}}\, \epsilon\, e^{-\beta\epsilon} d\epsilon$

$= \frac{\pi}{4}\left(\frac{L}{\hbar\pi\sqrt{A}}\right)^3 \int_0^\infty \frac{\epsilon^{3/2}}{e^{\beta\epsilon} - 1}\, d\epsilon$ Let $x = \beta\epsilon$

$= \frac{\pi}{4}\left(\frac{L}{\hbar\pi\sqrt{A}}\right)^3 \frac{1}{\beta^{5/2}} \int_0^\infty \frac{x^{3/2}}{e^x - 1}\, dx \Rightarrow E \propto T^{5/2}$

c. The pressure of a photon gas like this is $p = \frac{1}{3}\frac{E}{V}$

so $p \propto T^{5/2}$