4. *Quantum Mechanics* (Fall 2003)

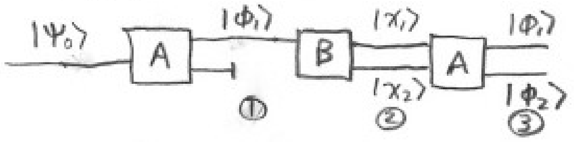
An operator A, corresponding to an observable α, has two normalized eigenfunctions ϕ_1 and ϕ_2, with distinct eigenvalues a_1 and a_2, respectively. An operator B, corresponding to an observable β, has normalized eigenfunctions χ_1 and χ_2, with distinct eigenvalues b_1 and b_2, respectively. The eigenfunctions are related by:

$$\phi_1 = (2\chi_1 + 3\chi_2)/\sqrt{13}$$
$$\phi_2 = (3\chi_1 - 2\chi_2)/\sqrt{13}.$$

An experimenter measures α to be $42\hbar$. The experimenter proceeds to measure β, followed by measuring α again. What is the probability the experimenter will measure α to be $42\hbar$ again?

We Know that the system starts in an eigenstate of A with eigenvalue $42\hbar$, but we don't Know if this is $|\phi_1\rangle$ or $|\phi_2\rangle$ so we will check both cases.

Case $|\phi_1\rangle$:



$$P_2(\chi_1;\psi_1) = |\langle\chi_1|\psi_1\rangle|^2 = |\langle\chi_1|\phi_1\rangle|^2 = \frac{4}{13}$$

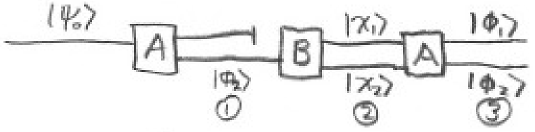
$$P_2(\chi_2;\psi_1) = |\langle\chi_2|\psi_1\rangle|^2 = |\langle\chi_2|\phi_1\rangle|^2 = \frac{9}{13}$$

$$P_3(\phi_1;\psi_2) = P_2(\chi_1;\psi_1)\,P_3(\phi_1;\chi_1) + P_2(\chi_2;\psi_1)\,P_3(\phi_1;\chi_2)$$

$$= \frac{4}{13}|\langle\phi_1|\chi_1\rangle|^2 + \frac{9}{13}|\langle\phi_1|\chi_2\rangle|^2$$

$$= \left(\frac{4}{13}\right)^2 + \left(\frac{9}{13}\right)^2 = \frac{16}{169} + \frac{81}{169} = \frac{97}{169}$$

Case $|\phi_2\rangle$:



$$P_2(\chi_1;\psi_1) = |\langle\chi_1|\psi_1\rangle|^2 = |\langle\chi_1|\phi_2\rangle|^2 = \frac{9}{13}$$

$$P_2(\chi_2;\psi_1) = |\langle\chi_2|\psi_1\rangle|^2 = |\langle\chi_2|\phi_2\rangle|^2 = \frac{4}{13}$$

$$P_3(\phi_2;\psi_2) = P_2(\chi_1;\psi_1)\,P_3(\phi_2;\chi_1) + P_2(\chi_2;\psi_1)\,P_3(\phi_2;\chi_2)$$

$$= \frac{9}{13}|\langle\phi_2|\chi_1\rangle|^2 + \frac{4}{13}|\langle\phi_2|\chi_2\rangle|^2$$

$$= \left(\frac{9}{13}\right)^2 + \left(\frac{4}{13}\right)^2 = \frac{81}{169} + \frac{16}{169} = \frac{97}{169}$$

So in either case the probability is $\frac{97}{169}$