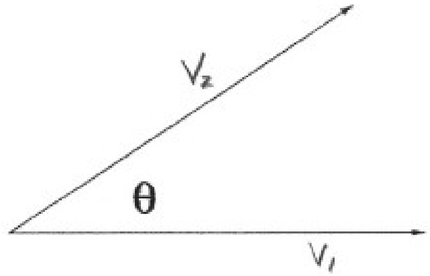
10. *Electricity and Magnetism* (Fall 2003)

Consider a wedge formed by two conducting half-planes, as depicted in the figure. One plane is maintained at electrostatic potential V_1 while the other is at V_2. What is the electrostatic potential in the region between the two half-planes?



We solve Laplace's equation in cylindrical coordinates

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \cancel{\frac{\partial^2\Phi}{\partial z^2}} = 0$$

We seek solutions of the form $\Phi(r,\phi) = R(r)\,Q(\phi)$

$$\frac{Q}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R}{r^2}\frac{\partial^2 Q}{\partial\phi^2} = 0$$

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2} = 0 \quad \left(\text{by multiplying by } \frac{r^2}{RQ}\right)$$

$$\Rightarrow r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \lambda R \quad \text{and} \quad \frac{\partial^2 Q}{\partial\phi^2} = -\lambda Q \quad (\text{by independence of variables})$$

When r ranges from 0 to ∞, only the $\lambda = 0$ eigenvalue is possible

$$\Rightarrow R(r) = A + B\ln(r) \quad \text{and} \quad Q(\phi) = C + D\phi$$

B.C. $\Phi(r=\infty) < \infty \Rightarrow B = 0 \Rightarrow \Phi(r,\phi) = C + D\phi$

$\Phi(r,0) = V_1 \Rightarrow C = V_1 \qquad \Phi(r,\theta) = V_2 \Rightarrow V_1 + D\theta = V_2$

$\Rightarrow D = \dfrac{V_2 - V_1}{\theta} \qquad$ Therefore $\Phi(r,\theta) = V_1 + \dfrac{V_2 - V_1}{\theta}\phi$