8. *Electricity and Magnetism* (Fall 2002)
Radiating Charges

(a) A point charge q under acceleration $\mathbf{a}(t)$ emits electromagnetic radiation. Give qualitative physical arguments why the radiated power, P, should be of the form $P = Bq^2a^2$, where B is a proportionality constant. Determine by dimensional analysis the dependence of B on fundamental physical constants. Explain how and why the exact expression for B differs from this estimate.

(b) A point charge q has mass m and is attached to a spring (of spring constant κ) hanging form a fixed support above an infinite horizontal **conducting** plane. The charge is set in motion with amplitude $A < h$, the equilibrium height of the charge above the conducting plane. Calculate its instantaneous radiating power.

a. $[P]=[Bq^2a^2]=[B][q^2a^2] \Rightarrow [B]=[P]/[q^2a^2]=\frac{(\mathrm{Kg\,m^2/s^2})/\mathrm{s}}{\mathrm{C^2 m^2/s^4}}=\frac{\mathrm{Kg\,s}}{\mathrm{C^2}}$

Now we need to look for appropriate physical constants

$[c]=\frac{m}{s}$, $\mu_0\epsilon_0=\frac{1}{c^2} \Rightarrow [\mu_0]=\left[\frac{1}{\epsilon_0 c^2}\right]=\frac{1}{[\epsilon_0]}\frac{s^2}{m^2}$

$F=\frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2} \Rightarrow [\epsilon_0]=\left[\frac{q^2}{Fr^2}\right]=\frac{C^2}{Nm^2}$

$\Rightarrow [\mu_0]=\frac{Nm^2}{C^2}\cdot\frac{s^2}{m^2}=\frac{kg\ m}{C^2}$

$\Rightarrow [B]=[\mu_0]\frac{s}{m}=[\mu_0/c] \Rightarrow B \propto \frac{\mu_0}{c}$

But really the Larmor formula is $P=\frac{\mu_0}{6\pi c}q^2a^2$ so $B=\frac{\mu_0}{6\pi c}$, which differs by a factor of $\frac{1}{6\pi}$ because dimensional analysis does not give constants of proportionality that are just geometric factors due to the angular distribution of radiation of $\sin^2(\theta)$.

b. The conducting plane effectively reflects the radiation without producing any additional net radiation, just as a mirror doesn't produce any additional light. The fields of the induced surface charge cancel in the lower half space and their radiation in the upper half space is exactly the reflected radiation. We know the fields cancel in the lower half space because the skin depth $\delta=\sqrt{\frac{2}{\mu\sigma\omega}} \to 0$ for $\sigma\to\infty$.

The motion of the charge is not harmonic because of the coulomb force with the image charge. We need to find the acceleration from a differential equation.

$$E=\frac{1}{2}K(z-h)^2+\frac{1}{4\pi\epsilon_0}\frac{q^2}{2z}+E_{rad}(t)+\frac{1}{2}m\dot{z}^2$$

$$0=\frac{dE}{dt}=K(z-h)\dot{z}+\frac{q^2}{4\pi\epsilon_0}\left(-\frac{\dot{z}}{z^2}\right)+\frac{\mu_0}{6\pi c}q^2\ddot{z}^2+m\dot{z}\ddot{z}$$

The instantaneous radiating power is $P=\frac{\mu_0}{6\pi c}q^2\ddot{z}^2(t)$ where $\ddot{z}^2(t)$ comes from solving the differential equation.