6. *Statistical Mechanics and Thermodynamics* (Fall 2002)

A gas of N highly relativistic, and non-interacting, spin 1/2 Fermions occupies a volume V at a temperature that is effectively equal to zero.

(a) Find the pressure on this gas.

(b) Based on the calculation you have just done, show what (extreme) inequality must be satisfied in order that the assumption of a temperature that is "effectively equal to zero" is justified.

(c) Suppose that the energy of the system due to gravitational self-attraction goes as $-AN^2V^{-1/3}$, where A is a constant. What does this and your result for the pressure imply about the stability of this system, assuming that gravitational attraction is what keeps it together?

a. $p = -\left(\frac{\partial E}{\partial V}\right)_S$ where $E = \int_0^\infty \epsilon f(\epsilon) \omega(\epsilon)\, d\epsilon$

T effectively zero $\Rightarrow f(\epsilon) = \Theta(\epsilon_F - \epsilon)$

$\Rightarrow E = \int_0^{\epsilon_F} \epsilon\, \omega(\epsilon)\, d\epsilon$

Let $g = 2$ be the spin-$\frac{1}{2}$ degeneracy

$\omega_n(n)\, dn = \frac{1}{8} g (4\pi n^2)\, dn$

$\omega_k(k)\, dk = \omega_n(n)\, dn$ if $k = \frac{\pi n}{L}$ and $dk = \frac{\pi\, dn}{L}$

$\Rightarrow \omega_k(k)\, dk = \frac{1}{8} g \left(4\pi \left(\frac{L}{\pi}\right)^2 k^2\right) \frac{L}{\pi} dk = \frac{V}{2\pi^2} g k^2 dk$

We find ϵ_F by the constraint that there are N occupied states.

$g \frac{1}{8}\left(\frac{4}{3}\pi n_F^3\right) = N \Rightarrow n_F = \left(\frac{3}{4}\frac{8N}{g\pi}\right)^{1/3} \Rightarrow n_F = \left(\frac{3N}{\pi}\right)^{1/3}$ since $g=2$

To find n_F, $\epsilon = \sqrt{p^2c^2 + m^2c^4} \cong pc$ if highly relativistic

$\Rightarrow \epsilon \cong pc = \hbar k c = \hbar c \frac{n\pi}{L}$

$\Rightarrow \epsilon_F = \frac{\hbar c \pi}{L} n_F$

So $\epsilon_F = \frac{\hbar c \pi}{L}\left(\frac{3}{\pi}\right)^{1/3} = \hbar c \left(3\pi^2 \frac{N}{V}\right)^{1/3}$

Therefore $E = \int_0^{\epsilon_F} \epsilon\, \omega(\epsilon)\, d\epsilon$

$\omega(\epsilon)\, d\epsilon = \omega_k(k)\, dk$ if $\epsilon = \hbar c k$ and $d\epsilon = \hbar c\, dk$

$\omega(\epsilon)\, d\epsilon = \frac{V}{\pi^2}(\hbar c)^{-3} \epsilon^2 d\epsilon$

$E = \int_0^{\epsilon_F} \frac{V}{\pi^2}(\hbar c)^{-3} \epsilon^3 d\epsilon$

$= \frac{V}{\pi^2}(\hbar c)^{-3} \frac{1}{4} (\hbar c)^4 \left(3\pi^2 \frac{N}{V}\right)^{4/3}$

$= \frac{3}{4} \hbar c \, (3\pi^2)^{1/3} N^{4/3} V^{-1/3}$

$p = -\left(\frac{\partial E}{\partial V}\right)_S = \frac{1}{4} \hbar c \, (3\pi^2)^{1/3} \left(\frac{N}{V}\right)^{4/3}$

b. $f(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} + 1}$ so we need $\mu \gg kT$

c. The system is stable if $\frac{\partial}{\partial V}(E + U) > 0$ $\quad U = -AN^2V^{-1/3}$

$\Leftrightarrow -\frac{1}{4}\hbar c (3\pi^2)^{1/3} \left(\frac{N}{\cancel{V}}\right)^{4/3} > -\frac{1}{3} A N^2 \cancel{V^{-4/3}}$

$\Leftrightarrow A > \frac{3}{4} \hbar c \, (3\pi^2)^{1/3} N^{-2/3}$