3. *Quantum Mechanics* (Fall 2002)

A charged particle of charge, q, and mass, m, is bound in a one-dimensional harmonic oscillator potential $V = \frac{1}{2}m\omega^2x^2$, where ω is the frequency of the oscillator. The system is then placed in an electric field E that is constant in space and time.

(a) Calculate the shift of the ground state energy to order E^2.

(b) What are the third and higher order (in E) shifts in the ground state energy? Give reasons for your answer.

Hint: If n labels the eigenstates of the unperturbed harmonic oscillator, then $\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n'}\delta_{n,n'-1} + \sqrt{n'+1}\delta_{n,n'+1}\right]$.

a. $V(x) = \frac{1}{2}m\omega^2x^2 + qEx$

$H = H^{(0)} + H'$ where $H^{(0)} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$ and $H' = qEx$

We Know the eigenvalues of $H^{(0)}$ are $E_n^{(0)} = (n+\frac{1}{2})\hbar\omega$.

$E_n = E_n^{(0)} + \Delta_n^{(1)} + \Delta_n^{(2)} + \dots$

$\Delta_n^{(1)} = H'_{nn} = \langle \psi_n^{(0)}|H'|\psi_n^{(0)}\rangle = qE\langle \psi_n^{\circ}|x|\psi_n^{\circ}\rangle = 0$

$\Delta_n^{(2)} = \sum_{m\neq n} \frac{|H'_{mn}|^2}{E_n^{\circ} - E_m^{\circ}} = \sum_{m\neq n} \frac{|\langle\psi_m^{(0)}|H'|\psi_n^{(0)}\rangle|^2}{(n+\frac{1}{2})\hbar\omega - (m+\frac{1}{2})\hbar\omega}$

$\Rightarrow \Delta_0^{(2)} = q^2E^2\sum_{m\neq 0} \frac{|\langle\psi_m^{(0)}|x|\psi_0^{(0)}\rangle|^2}{-m\hbar\omega}$

$= q^2E^2 \sum_{m\neq 0} \frac{\left|\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{m}\,\delta_{0,m-1} + \sqrt{m+1}\,\delta_{0,m+1}\right)\right|^2}{-m\hbar\omega}$

$= q^2E^2 \frac{\hbar}{2m\omega}\sqrt{1}/-m\hbar\omega$

$= -\frac{q^2E^2}{2m^2\omega^2}$

b. We can solve the eigenvalue problem exactly by completing the square in the potential.

$V(x) = \frac{1}{2}m\omega^2x^2 + qEx$

$\frac{2}{m\omega^2}V(x) = x^2 + \frac{2qE}{m\omega^2}x + \frac{q^2E^2}{m^2\omega^4} - \frac{q^2E^2}{m^2\omega^4}$

$= \left(x + \frac{qE}{m\omega^2}\right)^2 - \frac{q^2E^2}{m^2\omega^4}$

$V(x) = \frac{1}{2}m\omega^2\left(x + \frac{qE}{m\omega^2}\right)^2 - \frac{q^2E^2}{m^2\omega^4}$

Which is a simple harmonic oscillator with shifted equillibrium and an energy shift of $-q^2E^2/m^2\omega^4$.

Since the entire shift in ground state energy is second order in E, all higher order shifts are zero.