2. *Quantum Mechanics* (Fall 2002)

The Hamiltonian for a spinless charged particle in a magnetic field is

$$H = \frac{1}{2m}\left[\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r})\right]^2,$$

where the magnetic field $\mathbf{B}$ is related to the vector potential $\mathbf{A}$ by $\mathbf{B} = \nabla \times \mathbf{A}$. Here, e is the charge of the particle, m the mass, c the velocity of light and $\mathbf{p} = (p_x, p_y, p_z)$ is the momentum of the particle. Let $\mathbf{A} = -B_0 y\hat{x}$, corresponding to the magnetic field $\mathbf{B} = B_0\hat{z}$.

(a) Find the energy levels of the particle.

(b) Would the energy levels change if we chose $\mathbf{A}$ to be $\frac{B_0}{2}(-y\hat{x} + x\hat{y})$? Give reasons for your answer.

See Shankar page 210 and Abers section 6.1.3. (Landau Levels)

a. $H = \frac{1}{2m}\left[(p_x + \frac{e}{c}B_o y)^2 + p_y^2 + p_z^2\right] = H_{xy} + H_z$ where $H_z = \frac{p_z^2}{2m}$

$H_{xy} = \frac{1}{2m}\left[p_x^2 + 2\frac{e}{c}B_o p_x y + \frac{e^2}{c^2}B_o^2 y^2 + p_y^2\right]$ $\qquad \vec{r}\times\vec{p} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ x & y & z\\ p_x & p_y & p_z\end{vmatrix}$

$= \frac{1}{2m}\left[p_x^2 + 2\frac{e}{c}B_o(xp_y - L_z) + \frac{e^2}{c^2}B_o^2 y^2 + p_y^2\right]$ $\qquad \Rightarrow L_z = xp_y - yp_x$

$= \frac{1}{2m}\left[p_x^2 - 2\frac{e}{c}B_o L_z - \frac{e^2}{c^2}B_o^2 x^2 + \frac{e^2}{c^2}B_o^2 x^2 + 2\frac{e}{c}B_o x p_y + p_y^2 + \frac{e^2}{c^2}B_o^2 y^2\right]$

$= \frac{1}{2m}\left[p_x^2 - 2\frac{e}{c}B_o L_z - \frac{e^2}{c^2}B_o^2 x^2 + (p_y + \frac{e}{c}B_o x)^2 + \frac{e^2}{c^2}B_o^2 y^2\right]$

We replace p_y with $p_y' = p_y + \frac{e}{c}B_o x$ and L_z with $L_z' = xp_y' - yp_x$

So $L_z' = x(p_y + \frac{e}{c}B_o x) - yp_x = \frac{e}{c}B_o x^2 + L_z \Rightarrow L_z = L_z' - \frac{e}{c}B_o x^2$

$H_{xy} = \frac{1}{2m}\left[p_x^2 - 2\frac{e}{c}B_o L_z' + 2\frac{e^2}{c^2}B_o^2 x^2 - \frac{e^2}{c^2}B_o^2 x^2 + p_y'^2 + \frac{e^2}{c^2}B_o^2 y^2\right]$

$= \frac{1}{2m}\left[p_x^2 + \frac{e^2}{c^2}B_o^2 x^2 + p_y'^2 + \frac{e^2}{c^2}B_o^2 y^2 - 2\frac{e}{c}B_o L_z'\right]$

The terms with x and y are simple harmonic oscillators where

$\frac{1}{2}m\omega^2 = \frac{1}{2m}\frac{e^2}{c^2}B_o^2 \Rightarrow \omega = \frac{eB_o}{mc}$

But we need the simultaneous eigenstates of these and L_z
so we switch to the basis $|n_+, n_-\rangle$ defined with $a_\pm = \frac{1}{\sqrt{2}}(a_x \mp i a_y)$
and $n_\pm = a_\pm^\dagger a_\pm$ because then $n_x + n_y = n_+ + n_-$ and $L_z' = \hbar(n_+ - n_-)$.

$E_{xy} = (n_x + \frac{1}{2})\hbar\omega + (n_y + \frac{1}{2})\hbar\omega - \omega L_z$

$= (n_+ + n_- + 1)\hbar\omega - \hbar\omega(n_+ - n_-)$

$= (2n_- + 1)\hbar\omega$

$E = E_{xy} + E_z = (n + \frac{1}{2})\hbar\omega_c + \frac{\hbar^2 k_z^2}{2m}$ where $\omega_c = \frac{2eB_o}{mc}$, $n \in \{0, 1, 2, \dots\}$

b. $\vec{B} = \vec{\nabla}\times\vec{A} = \frac{B_o}{2}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial_x & \partial_y & \partial_z\\ -y & x & 0\end{vmatrix} = \frac{B_o}{2}(1+1)\hat{z} = B_o\hat{z}$

which is the same magnetic field, so the energy levels will be the same because of Gauge Invariance.