Dirac Notation

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1 Explanation

At the core of quantum mechanics is the concept of a quantum state that contains all the information there is about a particle. The mathematics of quantum mechanics utilizes a generalized vector notation to represent the quantum states.

A Hilbert space H is a vector space with an inner product that defines a norm that turns H into a complete metric space. A complete metric space is a metric space where all Cauchy sequences are convergent. A Cauchy sequence is a sequence such that the metric satisfies $\lim_{min(a_n,a_m)\to\infty} d(a_n, a_m) = 0$.

In quantum mechanics, we consider the continuously infinite dimensional complex Hilbert space of complex square integrable functions of the reals. In Dirac notation, we establish two identical Hilbert spaces of this type as a convenient way of keeping track of which vectors are complex conjugated and which are not. The non-complex conjugated vectors are called kets and are written $|\alpha\rangle$. The complex conjugated vectors reside in the dual space. They are called bras and are written $\langle\alpha|$.

Dirac notation is very useful because it establishes simple rules for combining bras, kets, and operators in meaningful ways. Note: Need an explanation for why it is complex - independent momentum values?

2 Wave functions

The most basic piece of information contained in a vector for a quantum state is the distribution of probabilities for the location of the particle in space. In fact, for simple cases i.e. without spin, this probability distribution and the probability distribution for momentum is all that the state vector contains. The distributions of both position and momentum are expressed in a complex valued function called the wave function. This is exactly the function that the state vector represents.

The state vector has a continuously infinite number of elements corresponding to the number of values of the wave function. For illustration we write the state vector as if it had only a countably infinite number of elements.

$$\langle\psi| = \begin{pmatrix} \psi(x=1) & \psi(x=2) & \cdots & \psi(x=n) \end{pmatrix}^*$$

In order to extract the probabilities from within the vector formalism we need a vector that will help us pick out a specific element of the state vector. We define the position bra as

$$\langle x| = \begin{pmatrix} 0 \; 0 \; \cdots \; 0 \; 1 \; 0 \; \cdots \; 0 \; 0 \end{pmatrix}$$

where the 1 is in the location specified by x. Therefore the inner product is

$$\langle x|\psi\rangle = \psi(x)$$

3 Outer Product

We can think of kets as column vectors and bras as row vectors. Then if we combine a ket and a bra in the form $|\alpha\rangle\langle\beta|$ we get a matrix. We call this infinite matrix the outer product. It can be thought of as an operator on the ket space because if you apply it to a ket you get another ket. It is worth noting that any operator on the ket space can be expressed as an infinite matrix. To express any operator as a matrix, you must first choose a basis of kets for the ket space. The matrix always depends on your choice of basis, each basis gives its own representation of the operator. Then the matrix simply takes the basis kets and reapportions them to make the transformed ket. The action on the basis kets is sufficient because

$$|\alpha'\rangle = X|\alpha\rangle = X \sum_n a_n|\alpha_n\rangle = \sum_n a_n X|\alpha_n\rangle = \sum_n a_n|\alpha_n'\rangle$$

which shows a transformed ket in terms of the transformed base kets.

4 Completeness Relation

Since the space is complete we have the completeness relation

$$\int |x\rangle\langle x| dx = I$$

which is an extremely useful way of expressing the identity operator. As we all know, all of algebra basically amounts to adding zero or multiplying by the identity, so this is identity is a large part of the algebra of Dirac notation.

It is useful to think of this as a sum of matrices where each term has a 1 in one location along the diagonal. By performing the entire sum, the diagonal fills out with ones to become the identity.

5 Inner Product

The inner product refers to a combination of a bra with a ket of the form $\langle\beta|\alpha\rangle$. Since the bras merely refer to complex conjugated forms of kets, if you switch their order, it complex conjugates both, so

$$\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$$

Using the identity operator we can see that

$$\langle\psi_1|\psi_2\rangle = \int_{-\infty}^{\infty} \langle\psi_1|x\rangle\langle x|\psi_2\rangle \, dx = \int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)dx$$

6 Associativity

Note that the same letter ψ is used for all bras and kets even though they may refer to different vectors and similarly X may refer to many different operators.

$\cancel{|\psi\rangle|\psi\rangle|\psi\rangle}$	$\cancel{|\psi\rangle|\psi\rangle\langle\psi|}$	$\cancel{|\psi\rangle|\psi\rangle X}$
$|\psi\rangle\langle\psi||\psi\rangle$	$\cancel{|\psi\rangle\langle\psi|\langle\psi|}$	$|\psi\rangle\langle\psi|X$
$\cancel{|\psi\rangle X|\psi\rangle}$	$\cancel{|\psi\rangle X\langle\psi|}$	$\cancel{|\psi\rangle XX}$
$\langle\psi||\psi\rangle|\psi\rangle$	$\langle\psi||\psi\rangle\langle\psi|$	$\langle\psi||\psi\rangle X$
$\langle\psi|\langle\psi||\psi\rangle$	$\cancel{\langle\psi|\langle\psi|\langle\psi|}$	$\cancel{\langle\psi|\langle\psi|X}$
$\langle\psi|X|\psi\rangle$	$\cancel{\langle\psi|X\langle\psi|}$	$\langle\psi|XX$
$\cancel{X|\psi\rangle|\psi\rangle}$	$X|\psi\rangle\langle\psi|$	$\cancel{X|\psi\rangle X}$
$X\langle\psi||\psi\rangle$	$\cancel{X\langle\psi|\langle\psi|}$	$\cancel{X\langle\psi|X}$
$XX|\psi\rangle$	$\cancel{XX\langle\psi|}$	XXX

Most of the remaining combinations are associative by the associativity of matrix and vector multiplication. There are three in particular that deserve further consideration.

$$\langle\psi||\psi\rangle|\psi\rangle \quad \langle\psi|\langle\psi||\psi\rangle \quad \langle\psi|X|\psi\rangle$$

The first two of these are semi-illegal because one order of operations is undefined while the other is defined. We know that it cannot be defined because looking at the first combination, the last two factors must be a ket that is parallel to the last ket, but there is no way of knowing the scale factor without knowing the first bra. Therefore the associative axiom is not entirely true. The third is the most interesting.

7 Adjoint

When we look at the expression

$$\langle\beta|X|\alpha\rangle$$

it actually is associative. We can apply the operator X to the left and get the same result for the inner product. The only catch is that the operator acts differently to the left than it does to the right. How can we find out how it acts? The best way is to define a new operator, called its adjoint or Hermitian conjugate that performs the same action, but as an operator from the left.

$$\langle X^\dagger \alpha| = \langle \alpha| X$$

Using this definition we can easily find the relationship between an operator and its adjoint.

$$(X)_{ij} = \langle i|X|j\rangle = \langle X^\dagger i|j\rangle = \langle j|X^\dagger i\rangle^* = \langle j|X^\dagger|i\rangle^* = (X^\dagger)^*_{ji}$$

$$\Rightarrow X^\dagger = (X^T)^*$$

This also implies directly that $(X^\dagger)^\dagger = X$.

8 Generator of Translations

Lets create an infinitesimal translation operator T such that

$$T(dx')|x'\rangle = |x' + dx'\rangle$$

We do not know what T is yet, but we can use our basic knowledge of what a translation is to write down some rules that the operator must obey. Here are four rules.

1. $\langle\alpha|\alpha\rangle = \langle\alpha|T^\dagger(dx')T(dx')|\alpha\rangle$
2. $T(dx'')T(dx') = T(dx' + dx'')$
3. $T(-dx') = T^{-1}(dx')$
4. $\lim_{dx'\to 0} T(dx') = I$

We can't prove that there is only one solution to these four requirements, but we can guess and check that

$$T(dx') = 1 - iK \cdot dx'$$

,where K is some Hermitian operator, is valid solution.

There is no known way to derive what K means physically, but some inspiration from classical mechanics points to momentum. It turns out that $K = p/\hbar$.

9 Canonical Commutation Relation

The commutator of two operators X and Y is defined to be

$$[X, Y] = XY - YX$$

and it is designed to be zero when the operators commute with each other. The canonical commutation relation refers to the commutator between the position and momentum operators.

$$[x, T(dx')]|x'\rangle = xT(dx')|x'\rangle - T(dx')x|x'\rangle$$

$$= x|x' + dx'\rangle - T(dx')x'|x'\rangle$$

$$= (x' + dx')|x' + dx'\rangle - x'|x' + dx'\rangle$$

$$= dx'|x' + dx'\rangle \cong dx'|x'\rangle$$

where the last step is only off by second order in an infinitesimal.

$$\Rightarrow x(1 - iK \cdot dx') - (1 - iK \cdot dx')x = dx'$$

$$\Rightarrow -i(xK \cdot dx' - K \cdot dx'x) = dx'$$

Let $dx' \to dx\hat{x}_j$

$$\Rightarrow xK_j dx - K_j dx x = idx\hat{x}_j$$

Cancel the dx and take the scalar product of both sides with $\hat{x}_i$.

$$\Rightarrow x_i K_j - K_j x_i = i\delta_{ij}$$

$$\Rightarrow [x_i, K_j] = i\delta_{ij}$$

10 Momentum Operator

Consider an infinitesimal translation on a ket $|\alpha\rangle$.

$$(1 - ip\,dx'_o/\hbar)|\alpha\rangle = T(dx'_o)|\alpha\rangle$$

$$= \int T(dx'_o)|x'\rangle\langle x'|\alpha\rangle \; dx'$$

$$= \int |x' + dx'_o\rangle\langle x'|\alpha\rangle \; dx'$$

Now let $x' \to x' - dx'_o$ so $dx' \to dx'$.

$$= \int |x'\rangle\langle x' - dx'_o|\alpha\rangle \; dx'$$

$$= \int |x'\rangle\psi_\alpha(x' - dx'_o) \; dx'$$

Now we Taylor expand the wave function about x'.

$$= \int |x'\rangle \left(\psi_\alpha(x') - \left.\frac{\partial \psi_\alpha}{\partial x'}\right|_{x'} dx'_o \right) \; dx'$$

$$= \int |x'\rangle \left(\langle x'|\alpha\rangle - dx'_o \frac{\partial}{\partial x'} \langle x'|\alpha\rangle \right) \; dx'$$

$$= |\alpha\rangle - dx'_o \int |x'\rangle \frac{\partial}{\partial x'} \langle x'|\alpha\rangle \; dx'$$

Comparing with the original expression and canceling the term $|\alpha\rangle$,

$$\Rightarrow (ip \, dx'_o/\hbar)|\alpha\rangle = dx'_o \int |x'\rangle \frac{\partial}{\partial x'} \langle x'|\alpha\rangle \; dx'$$

$$\Rightarrow p|\alpha\rangle = -i\hbar \int |x'\rangle \frac{\partial}{\partial x'} \langle x'|\alpha\rangle \; dx'$$

$$\Rightarrow \langle\beta|p|\alpha\rangle = -i\hbar \int \langle\beta|x'\rangle \frac{\partial}{\partial x'} \langle x'|\alpha\rangle \; dx'$$

$$= \int \psi^*_\beta(x') \left(-i\hbar \frac{\partial}{\partial x'} \right) \psi_\alpha(x') dx'$$

$$\Rightarrow p = -i\hbar \frac{\partial}{\partial x}$$