Blackbody Radiation

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1 Introduction

An object that absorbs all electromagnetic radiation incident upon it is called a **blackbody**. Kirchoff's Law states that for a body in thermal equilibrium, the emissivity is equal to the absorptivity. This is a simple consequence of the fact that a body cannot be in thermal equilibrium unless if it is absorbing and emitting equal amounts of energy. Kirchoff's Law tells us that blackbodies are also perfect emitters and therefore they have the simplest emission spectrum since it does not have any gaps in frequency where it is unable to emit radiation. All bodies in the real world have some quantizes emission spectrum and the blackbody spectrum is basically an envelope for the quantized energy spectrum.

A small hole in an object with a large cavity with low reflectivity produces the closest approximation to an ideal blackbody spectrum. The reason is that any light that enters the hole will have to reflect off the walls many times before it can escape through the hole, but there is only a very high probability that the light will be absorbed before this happens. Therefore, radiation leaving from the hole will almost certainly be from thermal emission. The concept here is that the reason things sometimes do not look like blackbodies is because the light you see has residual influences from the incident radiation.

The study of the blackbody spectrum is important because it was the first problem in physics to introduce quantization. Planck introduced his famous constant in order to resolve the ultraviolet catastrophe of the classical Rayleigh-Jeans blackbody spectrum.

Why do we count number of states if we are looking at thermal emission? There is a tricky argument based on detailed balance. If the body were to be in thermal equilibrium in the center of some cavity, then what temperature would have to prevail?

2 Energy Density

The density of states with a factor of two for polarization is

$$\rho(\omega)d\omega = 2\frac{1}{8}d^3n = 2\frac{1}{8}4\pi n^2 dn$$

and we have

$$\omega = ck = \frac{c\pi}{L} n \Rightarrow \rho(\omega)d\omega = \frac{V}{\pi^2 c^3}\omega^2 d\omega$$

Let ρ_u be the density of states in a unit volume,

$$\rho_u(\omega)d\omega = \frac{1}{\pi^2 c^3}\omega^2 d\omega$$

We want to find the average energy density due to radiation with angular frequency ω in a cavity at temperature T. So we need to average over all possible numbers of photons with the specified frequency, n. [1]

$$\frac{\text{energy}}{d^3\mathbf{k}\text{-volume}}(\mathbf{k}) = \frac{\text{energy}}{\text{particle}}(\mathbf{k}) \cdot \frac{\text{particles}}{\text{state}}(\mathbf{k}) \cdot \frac{\text{states}}{d^3\mathbf{k}\text{-volume}}(\mathbf{k})$$

Note that a state here refers to a specific standing wave mode, which corresponds to a specific frequency of radiation.

$$E(\omega)\ d\omega = \epsilon_1(\omega)n(\omega)\rho(\omega)\ d\omega$$

The energy density is found by setting the spatial volume $V \to 1$, which is what we did in ρ_u.

$$u(\omega)\ d\omega = \epsilon_1(\omega)n(\omega)\rho_u(\omega)\ d\omega$$

Note that ρ_u is just a scale factor for the $d\omega$ and at one specific frequency we actually have the simple expression $\epsilon(\omega) = \epsilon_1(\omega)n(\omega)$, which implies $\bar{\epsilon}(\omega) = \epsilon_1(\omega)\bar{n}(\omega)$ since $\epsilon_1(\omega) = \hbar\omega$ is just a constant.

$$\bar{u}(\omega)\ d\omega = \bar{\epsilon}(\omega)\rho_u(\omega)\ d\omega$$

Therefore our next step is to calculate the average energy of a standing wave of arbitrary frequency over all its excited states. First we will do it classically. In classical thermodynamics, energy is a continuous parameter, so we integrate over all positive energies. The probability of a specific excited mode occurring is given by the Boltzmann factor

$$P_n(\omega) \propto e^{-\beta\epsilon}$$

Therefore, the weighted average of energies is

$$\bar{\epsilon} = \frac{\int_0^\infty \epsilon\ e^{-\beta\epsilon}\ d\epsilon}{\int_0^\infty e^{-\beta\epsilon}\ d\epsilon}$$

$$= \frac{\int_0^\infty -\frac{\partial}{\partial\beta}e^{-\beta\epsilon}\ d\epsilon}{\int_0^\infty e^{-\beta\epsilon}\ d\epsilon}$$

[1] In Quantum Field Theory, photons are created and destroyed with operators analogous to the simple harmonic oscillator raising and lowering operators. So a system of photons in a cavity is analogous to a simple harmonic oscillator. One notable difference is that the ground state energy needs to be normalized away because $\epsilon = n\hbar\omega$ instead of $\epsilon = (n + 1/2)\hbar\omega$.

$$= -\frac{\partial}{\partial \beta} \ln \left(\int_0^\infty e^{-\beta \epsilon} \, d\epsilon \right)$$

$$= -\frac{\partial}{\partial \beta} \ln \left(-\frac{1}{\beta} \, e^{-\beta \epsilon} \Big|_0^\infty \right)$$

$$= -\frac{\partial}{\partial \beta} \ln \left(\frac{1}{\beta} \right)$$

$$= \frac{\partial}{\partial \beta} \ln(\beta) = \frac{1}{\beta} = kT$$

According to Eisberg and Resnick, this is a form of the equipartition theorem. However this gives us an energy density

$$\bar{u}(\omega; T) \, d\omega = \rho_u(\omega) \bar{\epsilon}(\omega) \, d\omega = \frac{1}{\pi^2 c^3} kT\omega^2 \, d\omega$$

which blows up to infinite quadratically for large frequencies. This clearly violates conservation of energy and the experimental data, so it was called the ultraviolet catastrophe.

The fix to this catastrophe was provided by Planck when he postulated that the energies of excited modes were quantized. We must think of the standing waves in the cavity as quantum simple harmonic oscillators with energy $\epsilon_n(\omega) = n\hbar\omega$. The probabilities of each excited mode are still given by the Boltzmann factor, but now we have to do an infinite sum over energies instead of an integral.

$$\bar{\epsilon} = \frac{\sum_{n=0}^\infty \epsilon_n(\omega) e^{-\beta \epsilon_n(\omega)}}{\sum_{n=0}^\infty e^{-\beta \epsilon_n(\omega)}}$$

$$= \frac{\sum_{n=0}^\infty -\frac{\partial}{\partial \beta} e^{-\beta \epsilon_n(\omega)}}{\sum_{n=0}^\infty e^{-\beta \epsilon_n(\omega)}}$$

$$= -\frac{\partial}{\partial \beta} \ln \left(\sum_{n=0}^\infty e^{-\beta \epsilon_n(\omega)} \right)$$

$$= -\frac{\partial}{\partial \beta} \ln \left(\sum_{n=0}^\infty e^{-\beta n \hbar \omega} \right)$$

$$= -\frac{\partial}{\partial \beta} \ln \left(\frac{1}{1 - e^{-\beta \hbar \omega}} \right)$$

$$= \frac{\partial}{\partial \beta} \ln \left(1 - e^{-\beta \hbar \omega} \right)$$

$$= \frac{1}{1 - e^{-\beta \hbar \omega}} e^{-\beta \hbar \omega} \hbar \omega$$

$$= \hbar \omega \frac{1}{e^{\beta \hbar \omega} - 1}$$

which is the Planck distribution function.

Inserting into the expression for energy density,

$$\bar{u}(\omega;T)\ d\omega = \rho_u(\omega)\bar{\epsilon}(\omega)\ d\omega = \frac{1}{\pi^2 c^3}\frac{\hbar\omega^3}{e^{\beta\hbar\omega}-1}\ d\omega$$

It is useful to introduce the dimensionless parameter $\eta = \beta\hbar\omega$.

$$\bar{u}(\omega;T)\ d\omega = \frac{\hbar}{\pi^2 c^3}\left(\frac{kT}{\hbar}\right)^4 \frac{\eta^3}{e^{\eta}-1}\ d\eta$$

This function of η is maximized for $\eta = \tilde{\eta} \approx 3$. Therefore we have

$$\tilde{\eta} = \beta_1\hbar\tilde{\omega_1} = \beta_2\hbar\tilde{\omega_2} \Rightarrow \frac{\tilde{\omega_1}}{T_1} = \frac{\tilde{\omega_2}}{T_2}$$

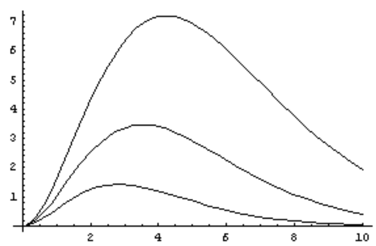
This is called Wien's displacement law.

We can get the total energy density by integrating over all frequencies.

$$\bar{u}_0(T) = \int_0^\infty \bar{u}(\omega;T)\ d\omega = \frac{\pi^2}{15}\frac{(kT)^4}{(c\hbar)^3}$$

This is not yet the Stefan-Boltzmann law, even though Reif calls it that. The Stefan-Boltzmann law says that the power radiated from a blackbody is σT^4.

However, we can now get a good picture of how the energy density vs. frequency curve morphs with changing temperature.



The peak shifts rightward linearly with temperature and upward with the fourth power of temperature.