Angular Momentum Algebra*

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1 Input

We will be going through the derivation of the angular momentum operator algebra. The only inputs to this mathematical formalism are the basic assumptions of quantum mechanics operators and the commutation relation between the components of angular momentum.

Axiom 1.1. $[J_i, J_j] = i\hbar \sum_k \epsilon_{ijk} J_k$ *or* $\mathbf{J} \times \mathbf{J} = i\hbar \mathbf{J}$

Since this is the only input, any operators that satisfy these commutation relations will obey the same algebra. The simple harmonic oscillator has a similar commutator and a similar algebra.

2 Setup

Before we start deriving the algebra, it is nice to do a little setup to make the work easier. First we define the square of the total angular momentum

Definition 2.1. $\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2$

and we pull a trick by defining the raising and lowering operators

Definition 2.2. $J_\pm = J_x \pm iJ_y$

Now we pre-compute all the commutation relations that we will need.

Lemma 2.3. $[\mathbf{J}^2, J_i] = 0$

Proof. We use the commutator identity $[A, BC] = [A, B]C + B[A, C]$

$$[\mathbf{J}^2, J_z] = [J_x^2 + J_y^2 + J_z^2, J_z] = [J_x^2, J_z] + [J_y^2, J_z] = -[J_z, J_x^2] - [J_z, J_y^2]$$
$$-[J_z, J_x]J_x - J_x[J_z, J_x] - [J_z, J_y]J_y - J_y[J_z, J_y]$$
$$= -i\hbar J_y J_x - i\hbar J_x J_y + i\hbar J_x J_y + i\hbar J_y J_x = 0$$

Similarly we can obtain the same result for J_x and J_y so we have $[\mathbf{J}^2, J_i] = 0$. □

*This lecture is mostly based on Sakurai section 3.5 with some additions from Shankar chapter 12.

Lemma 2.4. $[\mathbf{J}^2, J_\pm] = 0$

Proof.
$$[\mathbf{J}^2, J_\pm] = [\mathbf{J}^2, J_x \pm iJ_y] = [\mathbf{J}^2, J_x] \pm i[\mathbf{J}^2, J_y] = 0$$
□

Lemma 2.5. $[J_z, J_\pm] = \pm\hbar J_\pm$

Proof.
$$[J_z, J_x \pm iJ_y] = [J_z, J_x] \pm i[J_z, J_y] = i\hbar J_y \pm i(-i\hbar J_x)$$
$$= \hbar(\pm J_x + iJ_y) = \pm\hbar(J_x \pm iJ_y) = \pm\hbar J_\pm$$
□

Lemma 2.6. $[J_+, J_-] = 2\hbar J_z$

Proof.
$$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y] = i[J_y, J_x] - i[J_x, J_y]$$
$$= \hbar J_z + \hbar J_z = 2\hbar J_z$$
□

Finally we establish a maximal basis of commuting observables using $\mathbf{J}^2$ and J_z. We write the simultaneous eigenkets as $|a, b\rangle$ with the following eigenvalue equations:
$$\mathbf{J}^2|a, b\rangle = a|a, b\rangle \qquad \text{and} \qquad J_z|a, b\rangle = b|a, b\rangle$$

Note that we are not trying to say that a and b are integers, they are just the eigenvalues.

3 Applying $J_\pm$

What happens when we apply $J_\pm$? The operators $J_\pm$ were specially defined so that they would generate new simultaneous eigenkets from the simultaneous eigenkets. They do so in such a way that they form a ladder of states.

Theorem 3.1. $J_\pm|a, b\rangle = c_\pm|a, b \pm \hbar\rangle$

Proof.
$$J_z(J_\pm|a, b\rangle) = ([J_z, J_\pm] + J_\pm J_z)|a, b\rangle = (\pm\hbar J_\pm + J_\pm b)|a, b\rangle = (b \pm \hbar)(J_\pm|a, b\rangle)$$
$$\mathbf{J}^2(J_\pm|a, b\rangle) = J_\pm\mathbf{J}^2|a, b\rangle = a(J_\pm|a, b\rangle)$$

Therefore the state $J_\pm|a, b\rangle$ is the state with $\mathbf{J}^2$ eigenvalue a and J_z eigenvalue $b \pm \hbar$, unless if it is zero. Since the eigenstates of $\mathbf{J}^2$ and J_z form a basis, this state is determined up to a constant coefficient. Thus we have the result $J_\pm|a, b\rangle = c_\pm|a, b \pm \hbar\rangle$ □

4 Restriction on the J_z Eigenvalues

Theorem 4.1. *There are eigenvalues b_{max} and b_{min} of J_z such that $J_+|a, b_{max}\rangle = 0$ and $J_-|a, b_{min}\rangle = 0$*

Proof.

$$\langle a, b|\mathbf{J}^2 - J_z^2|a, b\rangle = \langle a, b|J_x^2 + J_y^2|a, b\rangle \geq 0$$

$$\Rightarrow a - b^2 \geq 0 \Rightarrow a \geq b^2$$

So there is a bound to the magnitude of b for a given a, which means that after some finite number of applications of the ladder operators, the zero state will arise. This is the statement of the theorem. Note that we are not saying $b_{max}^2 = a$ or $b_{min}^2 = a$. □

Theorem 4.2. *The eigenvalues are related by $a = b_{max}(b_{max} + \hbar)$*

Proof.

$$J_-(J_+|a, b_{max}\rangle) = J_-(0) = 0^1$$

$$(J_x^2 + J_y^2 - i(J_yJ_x - J_xJ_y))|a, b_{max}\rangle = 0$$

$$(\mathbf{J}^2 - J_z^2 + i[J_x, J_y])|a, b_{max}\rangle = 0$$

$$(\mathbf{J}^2 - J_z^2 - \hbar J_z)|a, b_{max}\rangle = 0$$

$$a - b_{max}^2 - \hbar b_{max} = 0$$

$$a = b_{max}(b_{max} + \hbar)$$

□

Theorem 4.3. *The eigenvalues are related by $a = b_{min}(b_{min} - \hbar)$*

Proof.

$$J_+(J_-|a, b_{min}\rangle) = J_+(0) = 0$$

$$(J_x^2 + J_y^2 + i(J_yJ_x - J_xJ_y))|a, b_{min}\rangle = 0$$

$$(\mathbf{J}^2 - J_z^2 - i[J_x, J_y])|a, b_{min}\rangle = 0$$

$$(\mathbf{J}^2 - J_z^2 + \hbar J_z)|a, b_{min}\rangle = 0$$

$$a - b_{min}^2 + \hbar b_{min} = 0$$

$$a = b_{min}(b_{min} - \hbar)$$

□

Theorem 4.4. $b_{min} = -b_{max}$

[1] Because all the operators are linear and linear operators on zero produce zero.

Proof. We are assuming a fixed a eigenvalue. By the last two theorems we find

$$\begin{aligned} b_{max}(b_{max}+\hbar) &= b_{min}(b_{min}-\hbar) \\ b_{max}^2 + b_{max}\hbar &= b_{min}^2 - b_{min}\hbar \\ b_{min}^2 - b_{max}^2 &= \hbar(b_{max}+b_{min}) \\ (b_{min}+b_{max})(b_{min}-b_{max}) &= \hbar(b_{max}+b_{min}) \\ \Rightarrow b_{max}+b_{min} = 0 \qquad &\text{or} \qquad b_{min}-b_{max} = \hbar \end{aligned}$$

Since the second is impossible, the first must be true, so $b_{min} = -b_{max}$. □

Theorem 4.5. *The eigenvalue b divided by $\hbar$ is either integer of half-integer.*

Proof. There must be a finite number n of steps on the ladder between the top and bottom states, so

$$\begin{aligned} b_{max} &= b_{min} + n\hbar \\ b_{max} &= -b_{max} + n\hbar \\ b_{max} &= \frac{n\hbar}{2} \\ \frac{b_{max}}{\hbar} &= \frac{n}{2} \end{aligned}$$

□

Remark 4.6. Orbital angular momentum only generates integer angular momentum because of the properties of the differential operator. But the commutation relation that we started with also allows half integer angular momentum. Nature does take advantage of this possibility with spin.

5 Redefining State Labels

It is convenient to label the states with integer or half integer quantities related to the eigenvalues rather than with the eigenvalues themselves. Let $j = b_{max}/\hbar$ and $m = b/\hbar$. Then we have $a = b_{max}(b_{max}+\hbar) = \hbar j(\hbar j + \hbar) = \hbar^2 j(j+1)$ and $b = \hbar m$. So the eigenvalue equations are now

$$\begin{aligned} \mathbf{J}^2|j,m\rangle &= \hbar^2 j(j+1)|j,m\rangle \\ J_z|j,m\rangle &= \hbar m|j,m\rangle \end{aligned}$$

6 Uniqueness

It is important to show that the ladder we created is unique or else we don't know if there are other states in unconnected ladders.

Theorem 6.1. *The ladder formed by applying J_- to $|a, b_{max}\rangle$ is unique.*

Proof. Any bounded ladder has b_{max} determined as in Theorem 4.2, so any other ladder must overlap with this ladder exactly, hence it is unique. □

7 Matrix Elements of $J_\pm$

Theorem 7.1. $J_\pm|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm 1)}|j,m\pm 1\rangle$

Proof.

$$\begin{aligned}\langle j,m|J_\pm^\dagger J_\pm|j,m\rangle &= \langle j,m|J_\mp J_\pm|j,m\rangle \\ &= \langle j,m|\mathbf{J}^2 - J_z^2 \mp \hbar J_z|j,m\rangle \\ &= \hbar^2[j(j+1) - m^2 \mp m] \\ &= \hbar^2[j(j+1) - m(m\pm 1)]\end{aligned}$$

But we also have

$$\langle j,m|J_\pm^\dagger J_\pm|j,m\rangle = \langle j,m\pm 1|c_\pm^* c_\pm|j,m\pm 1\rangle = |c_\pm|^2$$

$$\Rightarrow c_\pm = \hbar\sqrt{j(j+1)-m(m\pm 1)}$$

$$\Rightarrow J_\pm|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm 1)}|j,m\pm 1\rangle$$

□

Remark 7.2. If you need to find the expectation value of J_x or J_y, it may be easiest to express them in terms of $J_\pm$ and use the matrix element formula.