

1. *Quantum Mechanics* (Spring 2006)

An electron is at rest in a constant magnetic field pointing along the  $z$ -direction. The Hamiltonian is

$$H = -\boldsymbol{\mu} \cdot \mathbf{B} = g\mu_0 \frac{\mathbf{s}}{\hbar} \cdot \mathbf{B}$$

where  $\mathbf{B} = B_0 \hat{\mathbf{n}}_z$ . Since the electron is at rest, you can treat this as a two-state system. Let  $|\psi_{\pm}\rangle$  be the eigenstates of  $s_z$  with eigenvalues  $\pm\frac{\hbar}{2}$  respectively.

- (a) What are the eigenstates of the Hamiltonian in terms of  $|\psi_{\pm}\rangle$ , and what is the energy difference between them?
- (b) At time  $t = 0$  the electron is in an eigenstate of  $s_x$  with eigenvalue  $+\hbar/2$ . What is  $|\psi(0)\rangle$  in terms of  $|\psi_{\pm}\rangle$ ? Calculate  $|\psi(t)\rangle$  for any later time  $t$  in terms of these same two states.
- (c) For the state you calculated in part (b), what are the expectation values of the three components of the spin at any time  $t$ ?

2. *Quantum Mechanics* (Spring 2006)

The Hamiltonian for a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Let  $|\psi_n\rangle$ ,  $n = 0, 1, 2, \dots$ , be the usual energy eigenstates.

- (a) Suppose the system is in a state  $|\phi\rangle$  that is some linear combination of the two lowest states only:

$$|\phi\rangle = c_0 |\psi_0\rangle + c_1 |\psi_1\rangle$$

and suppose it is known that the expectation value of the energy is  $\hbar\omega$ . What are  $|c_0|$  and  $|c_1|$ ?

- (b) Choose  $c_0$  to be real and positive, but let  $c_1$  have any phase:  $c_1 = |c_1|e^{i\theta_1}$ . Suppose further that not only is the expectation value of  $H$  known to be  $\hbar\omega$ , but the expectation value of  $x$  is also known:

$$\langle \phi | x | \phi \rangle = \frac{1}{2} \sqrt{\frac{\hbar}{m\omega}}$$

What is  $\theta_1$ ?

- (c) Now suppose the system is in the state  $|\phi\rangle$  described above at time  $t = 0$ . That is,  $|\psi(0)\rangle = |\phi\rangle$ . What is  $|\psi(t)\rangle$  at a later time  $t$ ? Calculate the expectation value of  $x$  as a function of  $t$ . With what angular frequency does it oscillate?

### 3. Quantum Mechanics (Spring 2006)

A hydrogen atom is placed in a constant weak electric field of strength  $\varepsilon$ . Ignoring spin, what are the energies of the  $n = 1$  and  $n = 2$  levels including effects to first order in  $\varepsilon$  (but ignoring second order effects)?

Note: You may want to use some of the following:

Radial Wave Functions  $R_{nl}(r)$  ( $a$  is the Bohr radius):

$$\boxed{R_{10}(r) = \frac{1}{a^{3/2}} 2e^{-r/a} \quad R_{21}(r) = \frac{1}{a^{3/2}} \frac{1}{2\sqrt{6}} \frac{r}{a} e^{-r/2a}$$

$$R_{20}(r) = \frac{1}{a^{3/2}} \frac{1}{\sqrt{2}} \left(1 - \frac{r}{2a}\right) e^{-r/2a}}$$

Spherical Harmonics  $Y_l^m(\theta, \phi)$ :

$$\boxed{Y_0^0 = \frac{1}{\sqrt{4\pi}} \quad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta \quad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}}$$

An integral:

$$\boxed{\int_0^\infty x^n e^{-x/a} dx = a^{n+1} n!}$$

4. *Quantum Mechanics* (Spring 2006)

Consider the scattering of a beam of non-relativistic spin 0 particles by a repulsive spherical potential of depth  $V_0$  and radius  $a$  in three dimensions:

$$V(r) = \begin{cases} V_0, & \text{for } r < a; \\ 0, & \text{for } r > a. \end{cases}$$

Find the scattering cross section in the Born approximation.

5. *Quantum Mechanics* (Spring 2006)

A particle with mass  $m$  is confined to move on a circle of radius  $r$ . It is perturbed by a potential  $V(\theta) = a(1 + \cos(2\theta))$ .

- (a) What are the unperturbed energy levels?
- (b) Find the shift in the energy levels to first order in  $a$ .
- (c) Find the second order energy shift for all the states. Hint: Beware of the special care needed for some of the states.

6. *Statistical Mechanics and Thermodynamics* (Spring 2006)

Consider a gas of relativistic, conserved bosons. The relation between energy and momentum is

$$E = |\mathbf{p}|c$$

- (a) Derive the condition for Bose-Einstein condensation in three dimensions.
- (b) Does Bose-Einstein condensation occur in two dimensions? Justify your answer.
- (c) What is the highest dimension for which Bose-Einstein condensation does not occur?

7. *Statistical Mechanics and Thermodynamics* (Spring 2006)

- (a) In the case of a set of non-relativistic, noninteracting spin-1/2 fermions confined to two dimensions, what is the paramagnetic susceptibility at  $T = 0$ ? Give your answer in terms of the mass  $m$  of the fermions, the gyromagnetic ratio  $\gamma$ , the number density  $\sigma$ , and whatever fundamental constants (e.g.,  $\hbar$ ,  $c$ , ...) are necessary to completely specify the system.
- (b) Now, suppose that the external magnetic field is replaced by an effective field  $H_{\text{eff}}$ , where

$$H_{\text{eff}} = H + \Gamma \frac{M}{A}$$

where  $H$  is the external field and  $M/A$  is the induced magnetic moment per unit area. Above what threshold value of the parameter  $\Gamma$  is this two-dimensional system ferromagnetic at  $T = 0$ ?

8. *Statistical Mechanics and Thermodynamics* (Spring 2006)

- (a) A system consists of  $N$  particles, each of which can exist in two states, with energies  $\epsilon_0$  and  $\epsilon_1$ , respectively. Given that the total energy of this system is  $U$ , what is its entropy?
- (b) Obtain the expression for the entropy in the limit that  $N$  is large.
- (c) Now, give an expression for the temperature of this system, as a function of  $U$  and the energies of the single particle states. Does this expression have any properties that require some discussion?

Stirling's formula:  $n! \approx (\frac{n}{e})^n$ , when  $n$  is large.

9. *Statistical Mechanics and Thermodynamics* (Spring 2006)

A researcher claims that a particular substance in thermal equilibrium exhibits the following total-number-of-states function

$$\Omega(E) = c(E - E_0)^\alpha V^\gamma \exp\left(-\frac{g}{V}\right)$$

where  $E_0$ ,  $c$ ,  $\alpha$ ,  $\gamma$ , and  $g$  are positive coefficients independent of the energy  $E$ , the volume  $V$ , and the temperature  $T$ .

- (a) Find the equation of state for this substance.
- (b) What is the relationship between the average energy and the temperature?
- (c) Does this substance satisfy the third law of thermodynamics? Why?
- (d) What values should  $E_0$ ,  $c$ ,  $\alpha$ ,  $\gamma$ , and  $g$  take for this substance to behave as an ideal gas?

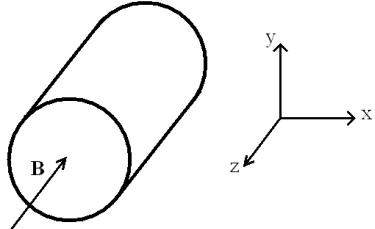
10. *Electricity and Magnetism* (Spring 2006)

An insulated, spherical, conducting shell of radius  $a$  is in a uniform electric field  $E_0$ . If the sphere is cut into two hemispheres by a plane perpendicular to the field, find the force required to prevent the hemispheres from separating

- (a) if the shell is uncharged;
- (b) if the total charge on the shell is  $Q$ .

11. Electricity and Magnetism (Spring 2006)

Consider a long solid cylinder made of uniform resistive material. The cylinder is in a region in which there is an applied magnetic field that is uniform and is directed along the axis of the cylinder. The magnetic field is time-dependent and it is oscillating with angular frequency  $\omega$ :  $\mathbf{B}(t) = B_z \cos \omega t \hat{z}$ . The length of the cylinder is  $L$  and its radius is  $R$  ( $R \ll L$ ). The resistivity of the cylinder material is  $\rho$ .



- (a) Calculate the current density  $\mathbf{j}(t)$  in the volume of the cylinder. Assume initially that you can ignore the self-inductance of the cylinder. Ignore end effects and the Hall effect.
- (b) For large values of  $\omega$  the effect of self-inductance cannot be ignored. Calculate the correction to the current density  $\Delta\mathbf{j}(t)$  due to the self-inductance of the cylinder in next order of  $\omega$ .
- (c) Give the condition on  $\omega$  such that the self-inductance of the cylinder can be ignored.

12. *Electricity and Magnetism* (Spring 2006)

Two point charges  $+Q_0$  and  $-Q_0$  are placed at opposite poles of a spherical balloon of initial radius  $R_0$ . The radius of the balloon is set to oscillate as follows:  $R(t) = R_0 + \rho \sin \omega t$ . Assume  $\rho\omega \ll c$ .

- (a) Determine the total power radiated by the oscillating balloon, if any, in terms of  $Q_0$ ,  $R_0$ ,  $\rho$ , and  $\omega$ . Show your work and explain your reasoning. (Note: if you are unable to write an expression for the total power radiated, explain how the total power radiated scales with each of the above variables.)
- (b) Suppose instead that charges are deposited on the balloon as described below. For each case, determine the ratio of the total power radiated by the oscillating balloon, if any, to the total power radiated in (a). Show your work and explain your reasoning.
  - (i) One point charge  $+Q_0$  is placed at a given point on the balloon. The radius of the balloon is set to oscillate as above.
  - (ii) A total charge  $+Q_0$  is deposited uniformly on the surface of the balloon. The radius of the balloon is set to oscillate as above.

13. *Electricity and Magnetism* (Spring 2006)

Consider an infinitely long filamentary current (i.e., a  $\delta$ -function) carrying a total current  $I$  along the  $z$ -direction.

- (a) Find the magnetic vector potential at a radial distance  $r$  from the current filament.

Now a non-relativistic particle of charge  $q$  and mass  $m$  is fired from a radial location  $d$  with velocity  $\mathbf{v}$  pointing in the radial direction, away from the current filament.

- (b) Evaluate the constants of the motion associated with the orbit of this particle.

- (c) Deduce the maximum radial distance reached by the particle.

- (d) What condition is required for the orbit size to be well-approximated by the usual Larmor radius expression?

14. *Electricity and Magnetism* (Spring 2006)

A plane, transverse electromagnetic wave of frequency  $\omega$  propagates through a scalar medium whose complex dielectric coefficient is given by

$$\epsilon(\omega) = 1 - \frac{a}{\omega(\omega + ib)}$$

where  $a$  and  $b$  are positive real constants.

- (a) What is the electrical conductivity of the medium?
- (b) What is the ratio of the magnitude of the material current density to the displacement current density in the medium?
- (c) Find the spatial damping coefficient of this wave (i.e., the imaginary part of  $k$ ) in the limit of small  $b$ .
- (d) Find the phase-shift between the electric and magnetic fields in the limit of small  $b$ .
- (e) Does this  $\epsilon(\omega)$  satisfy the required symmetry relation for general dielectric coefficients? Why?