1. *Quantum Mechanics* (Fall 2003)

Two identical spin-$\frac{1}{2}$ particles interact via the Hamiltonian

$$H = J(S_1^x S_2^x + S_1^y S_2^y + k S_1^z S_2^z) + \mu(S_1^z + S_2^z)B.$$

(a) Find the energy levels of this system assuming that the particles are in an anti-symmetric spatial wavefunction.

(b) Repeat for a symmetric spatial wavefunction.

2. *Quantum Mechanics* (Fall 2003)

A free particle of mass m, travelling with momentum p parallel to the z-axis, scatters off the potential

$$V = V_0 \left[\delta(\mathbf{r} - a\hat{\mathbf{z}}) - \delta(\mathbf{r} + a\hat{\mathbf{z}}) \right].$$

Compute the differential cross section, $d\sigma/d\Omega$ in the Born approximation.

3. *Quantum Mechanics* (Fall 2003)

Consider a particle moving in the potential

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2 & \text{if } x > 0 \\ \infty & \text{otherwise} \end{cases}$$

(a) What is the lowest energy eigenvalue?

(b) What is $\langle x^2 \rangle$?

4. *Quantum Mechanics* (Fall 2003)

An operator A, corresponding to an observable α, has two normalized eigenfunctions ϕ_1 and ϕ_2, with distinct eigenvalues a_1 and a_2, respectively. An operator B, corresponding to an observable β, has normalized eigenfunctions χ_1 and χ_2, with distinct eigenvalues b_1 and b_2, respectively. The eigenfunctions are related by:

$$\phi_1 = (2\chi_1 + 3\chi_2)/\sqrt{13}$$
$$\phi_2 = (3\chi_1 - 2\chi_2)/\sqrt{13}.$$

An experimenter measures α to be $42\hbar$. The experimenter proceeds to measure β, followed by measuring α again. What is the probability the experimenter will measure α to be $42\hbar$ again?

5. *Quantum Mechanics* (Fall 2003)

A sample of hydrogen atoms in the ground state is placed between the plates of a parallel plate capacitor. A voltage pulse is applied to the capacitor at $t = 0$ to produce a homogeneous electric field, ε, between the plates of:

$$\varepsilon = 0, \qquad (t < 0)$$

$$\varepsilon = \varepsilon_0 \exp(-t/\tau), \qquad (t > 0),$$

where τ is a constant. A long time compared to τ passes.

(a) To first order, calculate the fraction of atoms in the $2p$ $(m = 0)$ state.

(b) To first order, what is the fraction of atoms in the $2s$ state?

You may find the following helpful. The normalized radial wavefunctions of the hydrogen atom are:

$$R_{10}(r) = 2\left(\frac{Z}{a}\right)^{3/2} \exp\left(-\frac{Zr}{a}\right)$$

$$R_{20}(r) = \frac{1}{\sqrt{2}}\left(\frac{Z}{a}\right)^{3/2}\left(1 - \frac{Zr}{2a}\right) \exp\left(-\frac{Zr}{2a}\right)$$

$$R_{21}(r) = \frac{1}{2\sqrt{6}}\left(\frac{Z}{a}\right)^{5/2} r \exp\left(-\frac{Zr}{2a}\right)$$

where r is the radial coordinate, a is the Bohr radius, and $Z = 1$ for a hydrogen atom. The first spherical harmonics are:

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \qquad Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\theta) \qquad Y_{1\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin(\theta)\exp(\pm i\phi)$$

A useful integral may be:

$$\int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}$$

6. *Statistical Mechanics and Thermodynamics* (Fall 2003)

Consider a gas of non-conserved Bosons in three dimensions. The energy-verses-momentum relationship for each of these exotic particles is $E = Ap^2$.

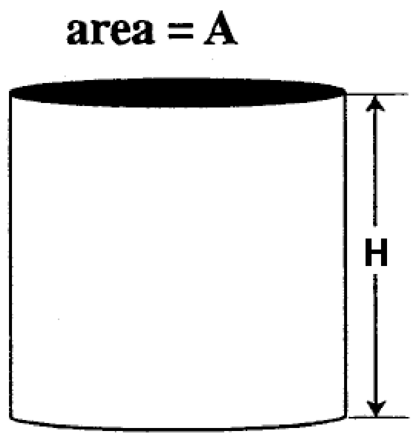
(a) Calculate the grand partition function of this gas. Assume a spin degeneracy factor of 1.

(b) Find the energy of this gas as a function of temperature. The energy goes as a simple power law in the temperature. What is the power?

(c) Find the pressure as a function of temperature. What power law describes the temperature dependence of the pressure?

7. *Statistical Mechanics and Thermodynamics* (Fall 2003)

A gas of noninteracting particles fills a cylindrical container that has cross-sectional area A and height H. Each particle has mass m, and is subject to the gravitational field at the surface of the Earth. The circular bottom and top of the container are parallel to the surface of the Earth. There are N particles in the container, and the temperature of the container is T.



(a) Find the partition function of the gas.

(b) What is the pressure of the gas at the top of the container?

(c) What is the pressure of the gas at the bottom of the container?

(d) Finally, what is the difference between the pressure at the bottom of the container and the pressure at the top of the container? Interpret the answer that you get.

8. *Electricity and Magnetism* (Fall 2003)

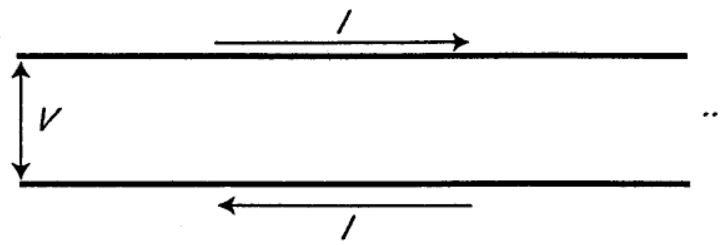
Consider a vacuum diode which is a parallel plate capacitor (in vacuum) with plate area A and plate separation d. The cathode plate, which is at $\phi = 0$, is heated as to thermionically emit electrons which then travel to the anode plate (at $\phi = V$) (this arrangement acts as a diode due to the fact that in reverse bias, no charges will flow). Assume a steady-state bias V and diode current I. You may model the electrons in the diode as a cold fluid with density $n(x)$ and velocity $v(x)$. You may assume that the electrons are born from the cathode with zero velocity.

(a) Find the 1-D potential distribution in the diode, $\phi(x)$. (Hint: Try a power law solution.)

(b) Find the diode current as a function of bias voltage V.

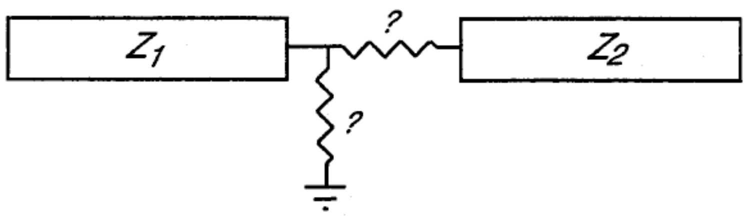
(c) What unphysical result is caused by the assumption that electrons are born from the cathode with zero velocity?

9. *Electricity and Magnetism* (Fall 2003)



(a) A two-wire transmission line has inductance L and capacitance C per unit length (and no resistance). Show that the impedance of this transmission line $Z = V/I$ is real and equal to $\sqrt{L/C}$ (Note: Assume AC signals are transmitted on the line, $I = I_o \exp(ikx - i\omega t)$).

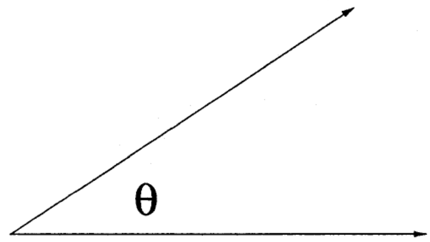
(b) Two long transmission lines are connected together. The first has impedance Z_1 and the second has impedance $Z_2 \neq Z_1$. A wave $V_i \exp(ikx - i\omega t)$ travels on the first transmission line and encounters the second. What are the relative amplitudes of the reflected and transmitted waves $(V_r/V_i, V_t/V_i)$?



(c) Reflection due to impedance mismatch between two transmission lines can be eliminated through adding series or parallel resistance between the lines. For the transmission lines in (b), how would you connect a resistor (and what is its values) in order to match the impedances and eliminate the reflected wave? (Consider both $Z_1 > Z_2$ and $Z_1 < Z_2$.)

10. *Electricity and Magnetism* (Fall 2003)

Consider a wedge formed by two conducting half-planes, as depicted in the figure. One plane is maintained at electrostatic potential V_1 while the other is at V_2. What is the electrostatic potential in the region between the two half-planes?



11. *Electricity and Magnetism* (Fall 2003)

The dispersion relation for a photon in an ionized plasma (in CGS units) is,

$$k^2c^2 = \omega^2 - 4\pi ne^2/m_e$$

where k is the photon wavenumber, $c = 3.0 \times 10^{10}$ cm/s, and ω is the radiation frequency in radians/s. Here, n is the electron number density, $e = 4.8 \times 10^{-10}$ esu is the electron charge, and $m_e = 9.11 \times 10^{-28}$ g is the electron mass.

(a) *Explain* why electromagnetic waves with frequencies below about 10 MHz can't be received from space on Earth.

(b) Pulsars are objects observed in our galaxy which regularly emit a short burst of electromagnetic waves containing a wide range of frequencies all at once. If a pulsar is located 1.0×10^{22} cm away and the density of electrons in the space between us and the pulsar is a uniform 0.01 cm^{-3}, what is the difference of the arrival times at Earth of the radiation emitted near 6 kHz compared to 10 kHz? (You may assume the measurement happens far enough above the Earth so that the effect in part (a) can be ignored. You may leave your answer as an expression without substituting the numbers.)

12. *Electricity and Magnetism* (Fall 2003)

(a) Consider and infinitely long electron beam with N electrons, a flat top radial profile with radius a, and velocity v_b. What is the force on an electron at the edge of the beam ($r = a$)?

(b) In reality no beam is infinitely long. Suppose the beam density has the form

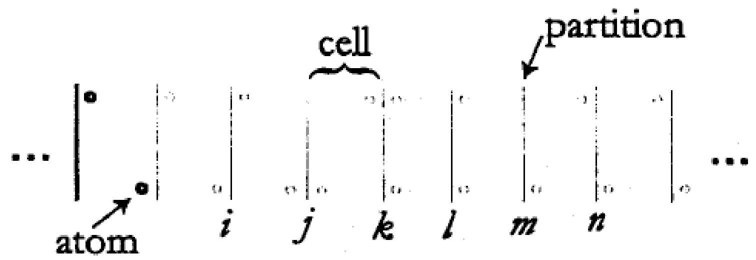
$$n_b = \frac{N}{\pi^{3/2}\sqrt{2}\sigma_z a^2} \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

for $r < a$ and is 0 for $r > a$ and its velocity is $\mathbf{v_b} = v_b\hat{\mathbf{z}}$. In the relativistic limit, what is the force (both in r and in z) for an electron at $r = a$ and at $z = 0$ and at $z = \sqrt{2}\sigma_z$? (Hint: One way to solve this problem is to start with the wave equations for the scalar and vector potentials, ϕ and $\mathbf{A}$, in the Lorentz gauge. Rewrite them in terms of the variables $x, y, \xi = z - v_b t$. Then simplify them in the limit $v_b \to c$. Use these equations to solve the problem.)

(c) For the electron beam at SLAC, $N = 2 \times 10^{10}$, $\sigma_z = 0.6$ mm, $a = 25$ μm, and the electrons have an energy of 50 GeV. Do the approximations used in part (b) hold for such a beam?

13. *Statistical Mechanics and Thermodynamics* (Fall 2003)

Consider a hypothetical system made up of N "partitions", a small section of which is shown in the figure below (the system is a closed ring, in order to eliminate end effects).



Each "cell" contains two atoms, one in the top half of the cell and one in the bottom half of the cell. Each atom occupies one of two positions in its half of the cell, to the left or to the right. The energies associated with an individual partition are given by the following rules: (i) Unless exactly two atoms are associated with a partition, the energy of that configuration is infinite (e.g. $\epsilon_k = \epsilon_m = +\infty$). (ii) If two atoms are on the same side of a partition, then the energy of that configuration is zero (e.g. $\epsilon_l = 0$). (iii) If two atoms are on opposite sides of a partition, then the energy of the configuration is ϵ (e.g. $\epsilon_i = \epsilon_n = \epsilon$).

(a) What are the energy levels possible for a system of N partitions and associated atoms? What is the degeneracy of each level? What is the canonical partition function for the system?

(b) Compute the free energy per particle in the thermodynamic limit and show that there is a discontinuity at a temperature T_c (i.e. the system exhibits a phase transition). Find T_c.

14. *Statistical Mechanics and Thermodynamics* (Fall 2003)

Consider a system of classical spins in d-dimensions which are confined to point at angles $\theta = 0, 2\pi/3, 4\pi/3$ in a plane, i.e. $\mathbf{s}_i = (\cos(\theta_i), \sin(\theta_i))$, with θ_i taking the three values above. The spins interact according to the Hamiltonian:

$$H = -J \sum_{\langle i,j \rangle} \mathbf{s}_i \cdot \mathbf{s}_j$$

where $\langle i, j \rangle$ are nearest neighbors. Using mean-field theory, find the critical temperature, T_c, below which the spins order.