Universal Field

Chris Clark April 12, 2008

1 Introduction

Let $|\Phi\rangle$ be a QFT state. The definition of the Schrodinger functional is

$$\Phi[\phi] \equiv \langle\phi|\,\Phi\rangle$$

where

$$\hat{\phi}\,|\phi\rangle = \phi\,|\phi\rangle$$

By taking the Hermitian conjugate of both sides,

$$\langle\phi|\,\hat{\phi}^\dagger = \phi^*\,\langle\phi|$$

2 One Particle

Using Hatfield equation (2.69), the QFT state corresponding to a universe containing a single particle with wave function ψ is

$$\left|\Phi_1^\psi\right\rangle = \int d^3x\ \psi(\mathbf{x},t)\hat{\phi}^\dagger(\mathbf{x},t)\,|\Phi_0\rangle$$

Therefore

$$\Phi_1^\psi[\phi] = \int d^3x\ \psi(\mathbf{x},t)\,\langle\phi|\,\hat{\phi}^\dagger(\mathbf{x},t)\,|\Phi_0\rangle$$

$$\Phi_1^\psi[\phi] = \int d^3x\ \psi(\mathbf{x},t)\phi^*(\mathbf{x},t)\,\langle\phi|\,\Phi_0\rangle$$

$$\Phi_1^\psi[\phi] = (\psi\cdot\phi^*)\Phi_0[\phi]$$

We notice that $\Phi_1^\psi[\psi] = \Phi_0[\psi]$ by the normalization of ψ. We also notice that any ϕ whose complex conjugate does not overlap with ψ will make the Schrodinger functional zero. Now, with our assumption that the Schrodinger functional represents the probability that some universal field is in the state given by the parameter, we should be able to extract the universal field itself. The universal field should be equal to whatever field ϕ maximizes the Schrodinger functional since that would make it the most likely field. For simplicity we will assume that the ground state functional does not affect the result, and hence divide it out. Therefore we will define the universal field ϕ_{univ} to be the field that maximizes $\Phi[\phi]/\Phi_0[\phi]$ subject to the constraint that $\int \phi(\mathbf{x},t)\phi^*(\mathbf{x},t)\ d^3x = N$, where N is the number of particles in the field. Using Lagrange multipliers for functional calculus, we see that ϕ_{univ} is the solution to

$$\frac{\delta}{\delta\phi^*(\mathbf{x})}\left(\frac{\Phi[\phi]}{\Phi_0[\phi]}\right) = \lambda\frac{\delta}{\delta\phi^*(\mathbf{x})}\int \phi(\mathbf{x}',t)\phi^*(\mathbf{x}',t)\ d^3x'$$

Now we must realize that we can treat ϕ and ϕ^* as independent variables. This follows from the fact that the real and imaginary parts of ϕ are independent. We can perform a change of variables to ϕ and ϕ^* retaining independence. [1] Therefore, ϕ_{univ} is the solution to

$$\frac{\delta}{\delta\phi^*(\mathbf{x})}\left(\frac{\Phi[\phi]}{\Phi_0[\phi]}\right) = \lambda\int \delta(\mathbf{x}'-\mathbf{x})\phi^*(\mathbf{x}',t)\ d^3x = \lambda\phi(\mathbf{x},t)$$

[1] See "Extremum Change of Variables" in the Derivations section of dfcd.net, and also see Quantum Theory of the Solid State by Lev Kantorovich pg 498.

3 Two Particles

Now let's consider the case of a universe containing two particles with non-overlapping wave functions ψ_1 and ψ_2.

$$\begin{aligned}
\Phi_2^{\psi_1\psi_2} &= \langle\phi| \int \psi_1(\mathbf{x}_1,t)\phi^\dagger(\mathbf{x}_1,t)\ d^3x_1 \int \psi_2(\mathbf{x}_2,t)\phi^\dagger(\mathbf{x}_2,t)\ d^3x_2\ \ |\Phi_0\rangle \\
&= \int \psi_1(\mathbf{x}_1,t)\phi^*(\mathbf{x}_1,t)\ \langle\phi| \int \psi_2(\mathbf{x}_2,t)\phi^\dagger(\mathbf{x}_2,t)\ d^3x_2\ \ |\Phi_0\rangle \\
&= \int \psi_1(\mathbf{x}_1,t)\phi^*(\mathbf{x}_1,t) \int \psi_2(\mathbf{x}_2,t)\phi^*(\mathbf{x}_2,t)\ d^3x_2\ \Phi_0[\phi]
\end{aligned}$$

Therefore,

$$\frac{\delta}{\delta\phi^*(\mathbf{x})}\left(\frac{\Phi_2^{\psi_1\psi_2}[\phi]}{\Phi_0[\phi]}\right) = \psi_1(\mathbf{x},t)(\psi_2\cdot\phi^*) + (\psi_1\cdot\phi^*)\psi_2(\mathbf{x},t)$$

So ϕ_{univ} is the solution to

$$\psi_1(\mathbf{x},t)(\psi_2\cdot\phi^*) + (\psi_1\cdot\phi^*)\psi_2(\mathbf{x},t) = \lambda\phi(\mathbf{x},t)$$

We can guess and check that the solution is

$$\phi_{univ}(\mathbf{x},t) = \psi_1(\mathbf{x},t) + \psi_2(\mathbf{x},t)$$

with $\lambda = 1$. This relies on the fact that ψ_1 and ψ_2 are non-overlapping. So it seems that ϕ_{univ} will tell you where the particles are in a given QFT state. It is very interesting that this constitutes a loss of information. When we constructed the state we knew that we had two localized particles, but after constructing the state, all we remember is the sum of the wave functions. In cases with no overlap we can assume that there are two distinct particles, but in cases of overlap, there will be indistinguishability issues inherent in the model. This is exactly analogous to the simple example of two ripples on a string travelling in opposite directions. When they meet, there is no correct answer to the question of whether they passed through each other or bounced off each other.[2]

4 Question

There are a few things that need to be looked into further. One obvious thing is the case of two particles with overlapping wave functions. Another is to check the assumption that we can divide by the ground state Schrodinger functional. It may be that this could affect the maximization. If not, it would still be nice to have some justification for why the ground state Schrodinger functional should be divided out.

[2] This example is from "An Interpretive Introduction to Quantum Field Theory" by Paul Teller.