Quantum Electrodynamics

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1 Introduction

Writing physical laws in the form of Lagrangians is one of the most important techniques in theoretical physics. Expressing physical laws in the form of a Lagrangian is like translating them into another language, but a special language that makes meaning more transparent. Lagrangians have the benefit of being scalar, which makes Lorentz invariance manifest. Furthermore, they make the procedure of identifying symmetries much simpler. They can also incorporate both field equations and force laws into a single expression. [1]
The Lagrangian density for free Dirac fields, corresponding to spin 1/2 is

$$\mathcal{L}_D = \bar{\psi}(i\hbar c\gamma^\mu \partial_\mu - mc^2)\psi$$

The QED Lagrangian Density is

$$\mathcal{L}_{QED} = \bar{\psi}(i\hbar c\gamma^\mu \partial_\mu - mc^2)\psi + e\hbar c A_\mu \bar{\psi}\gamma^\mu \psi - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and $A_\mu = (V/c, \mathbf{A})$ and $\bar{\psi} = \psi^\dagger \gamma^0$. The first term governs the propagation of free particles. The second term governs the interaction of particles with the electromagnetic field. The third term governs the propagation of the electromagnetic field. Minimal coupling is equivalent to introducing the interaction term in this Lagrangian density.

$$\mathcal{L}_{QED} = \bar{\psi}(i\hbar c\gamma^\mu(\partial_\mu - ieA_\mu) - mc^2)\psi - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}$$

This insertion is mandated by the requirement that the Lagrangian density be locally gauge covariant. That is also why the minimal coupling form of the derivative is sometimes called the covariant derivative.

2 Maxwell's Equations

In order to derive Maxwell's Equations from the QED Lagrangian density, all we have to do is vary with respect to A_ν and use the Euler-Lagrange equation. Recall the Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

Step 1. If we take $\phi = A_\nu$ in the Euler-Lagrange equation we have

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right)$$

Step 2. We evaluate the left hand side

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = e\hbar c \frac{\partial A_\mu}{\partial A_\nu} \bar{\psi}\gamma^\mu \psi = e\hbar c \delta^\nu_\mu \bar{\psi}\gamma^\mu \psi = e\hbar c \bar{\psi}\gamma^\nu \psi$$

because the first two terms are independent of A_ν. Recall that the way partial derivatives work is that derivatives of functions are independent of the original functions, which is why the first term is independent

[1] Is this correct?

of A_μ. [2]
Step 3. We evaluate the right hand side

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right)=\partial_\mu\left[-\frac{1}{4}\frac{\partial(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)}\right]$$

$$=-\frac{1}{4\mu_0}\partial_\mu\left[\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}F^{\alpha\beta}+F_{\alpha\beta}\frac{\partial F^{\alpha\beta}}{\partial(\partial_\mu A_\nu)}\right]$$

$$=-\frac{1}{4\mu_0}\partial_\mu\left[\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}F^{\alpha\beta}+F_{\alpha\beta}\frac{\partial(g^{\alpha\sigma}g^{\beta\rho}F_{\sigma\rho})}{\partial(\partial_\mu A_\nu)}\right]$$

$$=-\frac{1}{4\mu_0}\partial_\mu\left[\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}F^{\alpha\beta}+g^{\alpha\sigma}g^{\beta\rho}F_{\alpha\beta}\frac{\partial F_{\sigma\rho}}{\partial(\partial_\mu A_\nu)}\right]$$

$$=-\frac{1}{4\mu_0}\partial_\mu\left[\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}F^{\alpha\beta}+F^{\sigma\rho}\frac{\partial F_{\sigma\rho}}{\partial(\partial_\mu A_\nu)}\right]$$

$$=-\frac{1}{2\mu_0}\partial_\mu\left[\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}F^{\alpha\beta}\right]$$

$$=-\frac{1}{2\mu_0}\partial_\mu\left[\left(\frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\mu A_\nu)}-\frac{\partial(\partial_\beta A_\alpha)}{\partial(\partial_\mu A_\nu)}\right)F^{\alpha\beta}\right]$$

$$=-\frac{1}{2\mu_0}\partial_\mu\left[(\delta^\mu_\alpha\delta^\nu_\beta-\delta^\mu_\beta\delta^\nu_\alpha)F^{\alpha\beta}\right]$$

$$=-\frac{1}{2\mu_0}\partial_\mu(F^{\mu\nu}-F^{\nu\mu})$$

$$=-\frac{1}{\mu_0}\partial_\mu F^{\mu\nu}$$

Step 4. Combine the left and right hand sides and rearrange

$$e\hbar c\bar{\psi}\gamma^\nu\psi=-\frac{1}{\mu_0}\partial_\mu F^{\mu\nu}$$

$$\boxed{\partial^\mu F_{\mu\nu}=-e\mu_0\hbar c\bar{\psi}\gamma_\nu\psi}$$

Finally we have arrived at the inhomogeneous Maxwell's equations in covariant form. The other two equations result directly from the definition of $F_{\mu\nu}$.

3 Conserved Currents

Lagrangians of the form $\bar{\psi}\Gamma\psi$ for arbitrary Γ have a symmetry under phase shifts because a phase factor $e^{i\theta}$ and its complex conjugate in the same term cancel out. When we apply Noether's theorem to the QED Lagrangian with this symmetry, the conserved current equation produced is the well known continuity equation, from which local conservation of charge arises.

The actual expression for the conserved current is

$$j_\mu(x)\propto-\bar{\psi}\gamma_\mu\psi$$

which is consistent with the expression for the current that we found in Maxwell's equations. [3]

[2] Need to fully understand this.
[3] Question: How can we understand that this expression represents our common notion of charge currents?

4 Dirac Equation

The Dirac equation for the electron field can also be derived from this same Lagrangian. This time we vary with respect to $\bar{\psi}$. **Step 1.** If we take $\phi = \bar{\psi}$ in the Euler-Lagrange equation we have

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right)$$

Step 2. We evaluate the left hand side

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\hbar c \gamma^\mu \partial_\mu - mc^2)\psi + e\hbar c A_\mu \gamma^\mu \psi$$

Step 3. We evaluate the right hand side

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right) = 0$$

Step 4. We combine the left and right hand sides

$$\boxed{(i\hbar c \gamma^\mu \partial_\mu - mc^2)\psi = -e\hbar c A_\mu \gamma^\mu \psi}$$

5 Mini-Derivation

The definition of the variation of f for fixed x:

$$\bar{\delta} f(x) \equiv f'(x) - f(x) = [f'(x') - f(x)] - [f'(x') - f'(x)] = \delta f(x) - \partial_\mu f(x) \delta x^\mu$$

6 Annotated Bibliography

- Gauge Theories of Strong and Electroweak Interactions by Becher, Bohm, and Joos. Page 21 shows the QED Lagrangian density and states that Maxwell's equations and the Dirac equation can be derived from it. Page 74 shows the QCD Lagrangian density.

- Wikipedia: Lagrangian. Contains expressions of the Lagrangian densities in SI units.

- Spacetime and Geometry by Sean Carrol. Pages 42-43 do a similar derivation of Maxwell's equations, but starting with a non-QFT Lagrangian density.

- What is Spin? by Hans Ohanian in Am. J. Phys. Vol 54. No. 6 June 1986. Page 500. Explains that spin is actually momentum flow in the particle's wave field. Equation 20 is an interesting decomposition of the electron conserved current. Also gives an indication of where minimal coupling comes from two paragraphs before the conclusion.

- A First Book of Quantum Field Theory by Lahiri and Pal. Seems to be a simple introductory book. Mini-derivation comes from equation 2.41. Good derivation for gamma matrices is on page 50.