Group Theory

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1 Introduction

The goal of this paper is to organize some notes on group theory, mostly in an attempt to understand what was going on in the last paper on the Clebsch-Gordon decomposition.

2 The Kronecker Sum

Probably the biggest question that sticks out about the last paper is where did the "box product" definition come from. It turns out that the real name for this operator is the Kronecker sum and it is written as

$$A \oplus B \equiv A \otimes I_m + I_n \otimes B$$

where I represents the identity matrix of the appropriate dimension, which may be different in the two instances. The reason that the Kronecker sum is needed is because it has the following property

$$e^{A \oplus B} = e^A \otimes e^B$$

(See the Wikipedia article called Kronecker Product). This property tells us that the Kronecker sums of generators are the generators for the Kronecker product representation. We can see this more directly with an example. Consider the $2 \otimes 2$ representation of $SU(2)$, which consists of matrices of the form $U \otimes U$ where $U \in SU(2)$.[1] We will only show the proof for matrices U that are infinitely close to the identity matrix, but due to the fact $SU(2)$ is a Lie Group, this is sufficient.

$$\begin{aligned}
U \otimes U &= e^{i\alpha_i \sigma_i} \otimes e^{i\alpha_i \sigma_i} \\
&= (I + i\alpha_i \sigma_i) \otimes (I + i\alpha_i \sigma_i) \\
&= I \otimes I + i\alpha_i(\sigma_i \otimes I) + i\alpha_i(I \otimes \sigma_i) + O(\alpha^2) \\
&= I \otimes I + i\alpha_i(\sigma_i \oplus \sigma_i) \\
&= e^{i\alpha_i(\sigma_i \oplus \sigma_i)}
\end{aligned}$$

Therefore the generators of the $2 \otimes 2$ representation of $SU(2)$ are $\sigma_i \oplus \sigma_i$, where the is must be the same, but they are not summed over. To be sure that this is in fact a representation of $SU(2)$ we would have to show that these generators obey the same commutation relations as the generators of the defining representation.

3 Definitions

- The **order** of a finite group G, written $\#G$, is the number of elements in G. (Georgi-3)

- Let G be a group and M be a set. A map $\psi : G \times M \to M$ written as $\psi(a, m) = a \cdot m$ is a **group action** iff $a \cdot (b \cdot m) = (ab) \cdot m$ and $e \cdot m = m$ for all $a, b \in G$ and $m \in M$. (Sternberg-12) The motivation is that group actions allow us to establish a connection between abstract groups and symmetries on sets, which is the primary application of group theory.

- Let G_1 and G_2 be groups under the operations $\circ$ and $*$ respectively. A map $\phi : G_1 \to G_2$ is a **group homomorphism** iff $\phi(a \circ b) = \phi(a) * \phi(b)$ for all $a, b \in G_1$. (Sternberg-6) The motivation is that a group homomorphism indicates that G_2 has a subgroup isomorphic to G_1.

[1] If we allowed the two matrices to be different, then we would obtain a representation of $SU(2) \times SU(2)$ rather than $SU(2)$.

- Let G be a group and V be a vector space. A **group representation** D is a group homomorphism $D : G \to GL(V)$. (Wikipedia-Group Representation) That is a map $D : G \to GL(V)$ that satisfies $D(a)D(b) = D(ab)$ and $D(e) = I$ for all $a, b \in G$, where I is the identity transformation on V. The motivation is that group representations allow us to express group elements as matrices while ensuring that matrix multiplication behaves just like the group multiplication. Being able to express group elements as matrices is convenient because matrices can be rearranged by similarity transformations to explicitly show which subspaces are non-interfering under the action of the symmetries. [2]

- Let D and D' be two representations of the group G, mapping to $GL(V)$ and $GL(V')$ respectively. D and D' are **equivalent representations** or **similar representations**, written $D \sim D'$ iff there exists an invertible linear transformation $S : GL(V') \to GL(V)$ such that for all $a \in G$, $D'(a) = S^{-1}D(a)S$. (Georgi-5) If $V = V'$ we would say that there is a similarity transformation that relates the representations. The motivation is that equivalent representations can be used to find more convenient expressions for matrices that explicitly show which subspaces that are non-interfering under the action of the symmetry.

- The **dimension** or **degree** of a representation is the dimension of the vector space V of the group $GL(V)$ that the representation maps to. Physicists tend to use *dimension* (Georgi-3) while mathematicians tend to use *degree* (Sterberg-58). The motivation is that a representation of dimension or degree d can be expressed as a set of $d \times d$ matrices, so it gives us a simple way of characterizing representations by a simple property of their matrices. (Peskin-498) A given group can have representations of various degrees. Any group has the trivial representation of degree one.

- The element $D(a) \in D$ is called the **representation matrix** of the group element $a \in G$ in the representation D. (Gu-43) [3]

- A **Lie algebra** is a vector space V, along with a binary operation, called the Lie bracket, that satisfies bilinearity, anti-commutativity, and the Jacobi identity. (Wikipedia-Lie Algebra) [4]

- A **Lie Algebra associated to a Lie group** G is a Lie algebra whose vector space is the span of the generators of G and whose Lie bracket is the commutator. (Peskin-495) (Wikipedia-Lie Group)

- The **rank** of a Lie group is the number of generators of the group that simultaneously commute among themselves. (Kaku-54)

- A **Casimir operator** is an operator that commutes with all the generators of an algebra. (Kaku-55) For example, the group $O(3)$ has the Casimir operator $L^2 = L_1^2 + L_2^2 + L_3^2$, where the L_i are the generators.

- More terms to define: orbit, stabilizer/isotropy subgroup, transitive, coset, conjugacy class, morphism, invariant, cartesian product, fixed point set, index, character, generator, induced representation, restricted representation, Lie Group

- We also need explanations for fundamental, adjoint, defining, and regular representations.

4 Facts

- Gauge bosons are in the adjoint representation and all other particles are in the fundamental representation. (Zvi Bern)

[2] Sternberg defines group representations to be actions on V, but his definition is less clear and the Wikipedia definition is consistent with Georgi's definition.

[3] It seems that in one case *representation matrix* is used to mean *generator matrix*. "The representation matrices are given by the structure constants: $(t_G^b)_{ac} = if_{abc}$." (Peskin-499)

[4] A Lie algebra representation is different than a group representation.

- Mixed Product Property: $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ (Wikipedia-Kronecker Product)
- All representations of the same group have the same number of generators.
- Suppose that M decomposes into orbits under the action of G: $M = M_1 \cup \cdots \cup M_k$. Then we have a corresponding decomposition $r^M = r^{M_1} \oplus \cdots \oplus r^{M_k}$. (Sternberg-62)
- The relationship between characters and fixed point sets is: $\chi^{r^M}(a) = \#FP(a)$. (Sternberg-62)
- When we let a group act on functions on the group itself, the resulting representation contains all irreducible representations in its decomposition. (Sternberg-62)
- If r is an irreducible representation of G and s is an irreducible representation of H, then $r \otimes s$ is an irreducible representation of $G \times H$. (Sternberg-66)
- The number of distinct irreducible representations is equal to the number of conjugacy classes. (Sternberg-68)
- G is Abelian if and only if all its irreducible representations are one dimensional (Sternberg-71)
- Suppose that the group G has a commutative subgroup H. Then any irreducible representation of G has degree at most $\#G/\#H$.
- It is one of the axioms of quantum field theory that the fundamental fields of physics transform as irreducible representations of the Lorentz and Poincare groups. (Kaku-58)

5 Notation

- e is the identity element in the group G. (Sternberg-1)
- x^* is the adjoint of the matrix x. (Sternberg-7)
- $\#G$ is the number of elements in the group G. (Sternberg-13)
- $G \cdot m$ is the orbit of the point m under the action of G on M. (Sternberg-13)
- G_m is the isotropy group or stabilizer of the group G on the point m. (Sternberg-13)
- aG_m is the coset of element a in the group G_m. (Sternberg-14)
- G/G_m is the set of cosets of the group G. (Sternberg-14)
- $G \times M$ is the cartesian product of the sets G and M. (Sternberg-24)
- $FP(a)$ is the fixed point set of the element a in the group G. (Sternberg-25)
- $r \sim r'$ means that r and r' are equivalent or similar representations. (Sternberg-49)
- $\mathrm{Hom}_G(V_1, V_2)$ is the vector space of all linear maps from V_1 to V_2. (Sternberg-55)
- $\mathcal{F}(M)$ is the vector space of all complex-valued functions on the set M. (Sternberg-60)

6 Annotated Bibliography

- Group Theory and Physics by Sternberg - Contains good definitions for group theory
- Lie Algebras in Particle Physics by Georgi - Defines group theory terms in physicist's language
- An Introduction to Quantum Mechanics by Peskin and Schroeder - Contains a small amount of vague information on group theory
- Problems & Solutions in Group Theory for Physicists by Gu - Used for definition of representation matrix
- Quantum Field Theory: A Modern Introduction by Kaku - Used for definition of rank of a Lie Group
- Wikipedia "Kronecker Product" - States properties and defines Kronecker sum