Formulations of QFT

Chris Clark April 12, 2008

1 What are the formulations?

There are three basic formulations of constructive quantum field theory (cf. Bogoliubov,N.N., Shirkov,D.V.: Introduction to the theory of quantized fields, Wiley, New York, 1980).

> 1. Canonical formulation. Quantum fields are operator-valued fields on the Minkowski space-time Rd,1 that satisfy the canonical commutation relations and solve the classical Hamiltonian equations. For interacting fields the equations are non-linear partial differential equations on Rd,1. Unfortunately for d > 3, the relativistic irreducible quantum fields, which satisfy the canonical commutation relations, are free by default (cf.[3]). Even the d = 3 case is troublesome. The simplest non-linearity in the perturbation theory is the 4 interaction. Yet for d = 3 renormalization screens out the perturbation (cf.[7]).
>
> 2. Feynman formulation. [Functional integral/Path integral] The quantum propagators of classical fields are Feynman integrals over classical histories on the Minkowski space-time Rd,1. Since 1960s the prevalent approach is the Lagrangean Feynman- Kac infinite-dimensional integral over the space of histories on the euclidean space Rd+1 with the aposteriory analytic continuation to the real time. This approach of K.Symanszik and E.Nelson has culminated in the work of J.Glimm and A.Jaffe [10]. However, its application to interacting fields in the space dimensions d > 2 is still open.
>
> 3. Functional Schrodinger formulation. The quantum states are functionals on the phase space propagated by the evolution operator of a linear functional differential Schrodinger equation. For quite some time the functional Schrodinger formulation has been presumed mathematically unreasonable. Yet important analytic techniques for functional differential equations have been developed in the P. Kree seminar at the Institut Henri Poincare in Paris during 70s.

The Feynman rules have been derived in the first two pictures. Peskin and Schroeder chapter 9, page 275 says

> ... we introduce in this chapter an alternative method of deriving the Feynman rules for an interacting quantum field theory: the method of functional integration.

Chapter 4 derives the Feynman rules in the canonical formulation and chapter 9 derives them in the Feynman/functional integral formulation as stated in this quotation.
In the paper "Precanonical quantization and the Schrodinger wave functional", Igor Kanatchikov defines a fourth formulation which he calls "Precanonical quantization". This seems to be the same formulation that we have been developing. However to me it seems that our formulation is just a more explicit form of the Schrodinger formulation.

2 Why are multiple formulations?

I think that the various formulations correspond to different ways of enforcing that the underlaying field obeys the desired field equation (the Klein-Gordon equation for our scalar field theory model).

$$1. \quad \hat{\mathcal{O}}\hat{\phi}(\mathbf{p}, t) = 0$$

$$2. \quad \langle \phi_b(\mathbf{x}) |\, e^{-iHt} \,| \phi_a(\mathbf{x}) \rangle = \int \mathcal{D}\phi \exp\left[i \int_0^t d^4x \; \mathcal{L} \right]$$

$$3. \quad \hat{H}\left|\Phi\right\rangle = i\hbar\partial_t\left|\Phi\right\rangle$$

where $\hat{\mathcal{O}}$ is the Klein-Gordon equation operator, $\mathcal{L}$ is the Klein-Gordon Langrangian, and $\hat{H}$ is the Klein-Gordon Hamiltonian.

3 Feynman Rules

Following the development in Aitchison and Hey chapter 6, I created a flowchart showing how to derive the Feynman rules in the canonical formulation.
The Tomonaga-Schwinger equation (simplified).

$$i\frac{d}{dt}\left|\psi(t)\right\rangle_I = \hat{H}_i'(t)\left|\psi(t)\right\rangle_I \quad (6.25)$$

The definition of the scattering operator $\hat{S}$.

$$\hat{S}\left|i\right\rangle \equiv \left|\psi(\infty)\right\rangle_I \quad (6.27)$$

The Dyson series expansion from Quantum Mechanics

$$U(t) = 1 + \int_{-\infty}^{t} (-i\hat{H}_I'(t_1)\ dt_1 + \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\ (-i\hat{H}_I'(t_1))(-i\hat{H}_I'(t_2)) + \cdots \quad (6.35)$$

These all lead to the Dyson series expansion of the $\hat{S}$ operator (6.42).
Now if we take the matrix element of $\hat{S}$ and insert the interaction terms from the Hamiltonian we get an expression for the transition amplitude,

$$\left\langle p_a', p_b' \middle| \hat{S} \middle| p_a, p_b \right\rangle$$

Now defining the invariant amplitude by

$$S_{fi} = \delta_{fi} + i(2\pi)^4\delta^4(p_f - p_i)\mathcal{M}_{fi} \quad (6.102)$$

and using Wick's theorem we discover the Feynman rules for determining $\mathcal{M}$.
So where does quantum field theory enter into all of this? Most of the assumptions going in are either definitions or quantum mechanics. I think the only place that QFT enters is through the insertion of the interaction terms. This makes sense because in the interaction terms consist of $\hat{\phi}$ field operators that satisfy special commutation relations that were generated by the assumption $\hat{\mathcal{O}}\hat{\phi} = 0$ (See my third lecture of Winter quarter). So it stands to reason that if the Schrodinger formulation enforces this same constraint, then the Feynman rules should follow from it to.
Note that the non-interaction part of the Hamiltonian is not explicitly inserted. It still does affect the result, however, because the $\hat{\phi}$ operator is defined so that it satisfies the field equation that corresponds to the non-interaction part of the Hamiltonian, so the information is still there.