Field Energy

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In this paper we will show that if ϕ is a Klein-Gordon energy eigenstate, then the Klein-Gordon Hamiltonian applied to the field ϕ gives

$$\hat{H}\phi(\mathbf{x}) = \hbar\omega_\phi(\mathbf{x})\phi(\mathbf{x})$$

where

$$\omega_\phi(\mathbf{x}) \equiv \frac{\int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}\omega(\mathbf{q})\tilde{\phi}(\mathbf{q})}{\int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}\tilde{\phi}(\mathbf{q})}$$

where

$$\omega(\mathbf{q}) \equiv \frac{1}{\hbar}\sqrt{q^2c^2 + m^2c^4}$$

First of all, we know that for the Klein-Gordon equation,

$$\hat{H}^2 = \hat{p}^2c^2 + m^2c^4 \doteq -\hbar^2c^2\nabla^2 + m^2c^4$$

When we apply this to a plane wave state $\phi_\mathbf{q} = e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}e^{i\omega_0 t}$ we find

$$\hat{H}^2\phi_\mathbf{q} = (-\hbar^2c^2\nabla^2 + m^2c^4)e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}e^{i\omega_0 t}$$

$$= (-\hbar^2c^2(-q^2/\hbar^2) + m^2c^4)e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}e^{i\omega_0 t}$$

$$= \hbar^2\omega^2(\mathbf{q})\phi_\mathbf{q}$$

We need to check if plane waves are in fact energy eigenstates and not just eigenstates of $\hat{H}^2$. We can do this by taking the Taylor expansion of the square root in the Hamiltonian. We use the following Taylor series

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots$$

Therefore

$$\hat{H} = \sqrt{\hat{p}^2c^2 + m^2c^4} = mc^2\sqrt{1 + \frac{\hat{p}^2}{m^2c^2}}$$

$$= mc^2\left(1 + \frac{1}{2}\left(\frac{\hat{p}^2}{m^2c^2}\right) - \frac{1}{8}\left(\frac{\hat{p}^2}{m^2c^2}\right)^2 + \cdots\right)$$

$$\doteq mc^2\left(1 - \frac{1}{2}\frac{\hbar^2\nabla^2}{m^2c^2} - \frac{1}{8}\frac{\hbar^2\nabla^4}{m^4c^4} + \cdots\right)$$

Now since the plane wave $\phi_\mathbf{q}$ is an eigenstate of the ∇^2 operator, it is an eigenstate of this Hamiltonian. So for a plane wave $\phi_\mathbf{q}$ we have

$$\hat{H}\phi_\mathbf{q} = E_\mathbf{q}\phi_\mathbf{q}$$

$$\hat{H}^2\phi = E_\mathbf{q}^2\phi_\mathbf{q} = \hbar^2\omega^2(\mathbf{q})\phi_\mathbf{q}$$

Therefore

$$E_\mathbf{q} = \hbar\omega(\mathbf{q})$$

We chose only the positive root because the Hamiltonian is the positive square root, so its eigenvalues can never be negative.

Now, for a general energy eigenstate ϕ, we Fourier expand to find

$$\hat{H}\phi(\mathbf{x}) = \int \frac{d^3q}{(2\pi\hbar)^3}\hat{H}\left(e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}\right)\tilde{\phi}(\mathbf{q})$$

$$= \int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar} \hbar\omega(\mathbf{q})\tilde{\phi}(\mathbf{q})$$

Multiplying and dividing by ϕ,

$$= \hbar \frac{\int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar} \omega(\mathbf{q})\tilde{\phi}(\mathbf{q})}{\int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar} \tilde{\phi}(\mathbf{q})} \phi(\mathbf{x})$$

So we have in the position representation,

$$\hat{H}\phi(\mathbf{x}) = \hbar\omega_\phi(\mathbf{x})\phi(\mathbf{x})$$