Commutators in Group Theory

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1 Introduction

In this paper we will prove the following theorem.

> **Theorem** Let $D_1 : G \to GL(V_1)$ be a representation of a Lie group G with generators T_i i.e. for all $g \in G$, $D_1(g) = e^{\alpha_i T_i}$ for some real parameters α_i. If we are given a set of generators T_i' with the same commutation relations as the set T_i, then there exists a representation D_2 of G that maps to the Lie group generated by the set T_i'.

The definition of *representation* given in Georgi's book is the following: A **representation** of G is a mapping, D of the elements of G onto a set of linear operators with the following properties: (1) $D(e) = I$, where I is the identity operator on the space on which the linear operators act and (2) $D(g_1)D(g_2) = D(g_1 g_2)$, in other words the group multiplication law is mapped onto the natural multiplication in the linear space on which the linear operators act.

The proof depends on the Baker-Campbell-Hausdorff formula which says

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]-[Y,[X,Y]])+\cdots}$$

where the ellipsis represents higher order multiple commutators. [1]

2 Proof

The first step is to define a Lie group homomorphism Φ. Let $GL(V_2)$ be the linear space generated by the generators T_i'. We define $\Phi : D_1(G) \to GL(V_2)$ to take $D_1(g)$ to the element in $GL(V_2)$ whose parameters are the same as the parameters of $D_1(g)$, so if $D_1(g) = e^{\alpha_i T_i}$, then $\Phi[D_1(g)] = \Phi[e^{\alpha_i T_i}] = e^{\alpha_i T_i'} \in GL(V_2)$. So Φ basically converts T_i to T_i' (as long as the parameter is of the form $e^{\alpha_i T_i}$). We now check that this defines a group homomorphism.

$$\Phi[D_1(g)D_1(g')] = \Phi[e^{\alpha_i T_i} e^{\alpha_i' T_i}]$$

$$= \Phi[e^{\alpha_i T_i + \alpha_i' T_i + \frac{1}{2}[\alpha_i T_i, \alpha_i' T_i] + \frac{1}{12}([\alpha_i T_i,[\alpha_i T_i, \alpha_i' T_i]] - [\alpha_i' T_i,[\alpha_i T_i, \alpha_i' T_i]]) + \cdots}]$$

$$= \Phi[e^{\tilde{\alpha}_i T_i}] = e^{\tilde{\alpha}_i T_i'}$$

where $\tilde{\alpha}_i$ is the complicated function of α_i and α_i' formed by all the commutators. Now because of the assumption of the same commutation relations we can unwrap the definition of $\tilde{\alpha}_i$ in the exact same way that we wrapped it up, but now the generators all have primes

$$= e^{\alpha_i T_i' + \alpha_i' T_i' + \frac{1}{2}[\alpha_i T_i', \alpha_i' T_i'] + \frac{1}{12}([\alpha_i T_i',[\alpha_i T_i', \alpha_i' T_i']] - [\alpha_i' T_i',[\alpha_i T_i', \alpha_i' T_i']]) + \cdots}$$

$$= e^{\alpha_i T_i'} e^{\alpha_i' T_i'}$$

$$= \Phi[e^{\alpha_i T_i}]\Phi[e^{\alpha_i' T_i}]$$

$$= \Phi[D_1(g)]\Phi[D_1(g')]$$

This proves that Φ is a group homomorphism.

Now define the mapping D_2 by $D_2(g) = \Phi[D_1(g)]$. Then we have $D_2(e) = \Phi[D_1(e)] = e^0 = I$ and

$$D_2(gg') = \Phi[D_1(gg')] = \Phi[D_1(g)D_1(g')] = \Phi[D_1(g)]\Phi[D_1(g')] = D_2(g)D_2(g')$$

[1] From Wikipedia "Baker-Campbell-Hausdorff Formula".