Clebsch–Gordon Decomposition

Chris Clark   February 8 2008

Definitions

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$J_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

```
Sigma := {σ0, σ1, σ2, σ3}

Needs["LinearAlgebra`MatrixManipulation`"]

KroneckerProduct[a_?SquareMatrixQ, b_?SquareMatrixQ] := BlockMatrix[Outer[Times, a, b]]

BoxProduct[a_?SquareMatrixQ, b_?SquareMatrixQ] :=
 KroneckerProduct[a, σ0] + KroneckerProduct[σ0, b]

ZeroVecQ[v_] := Module[{i},
  For[i = 0, i <= Length[v], i = i + 1, If[Extract[v, i] ≠ 0, Return[0]]]; Return[1];]

TruncateMatrix[m_] :=
 Module[{i}, For[i = 0, ZeroVecQ[Extract[m, Length[m] - i]] ≠ 0, i = i + 1, i = i];
  Return[Drop[m, -(i)]];]
```

Linearly Independent Generators in the 2x2 Representation

First we generate a table of all possible box products of the sigma matrices. We expect the generators of the 2x2 representa–tion to be found in this set, or linear combinations of matrices in this set.

```
MatrixForm[
 Table[Table[BoxProduct[Extract[Sigma, j], Extract[Sigma, i]], {i, 2, 4}], {j, 2, 4}]]
```

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \\ 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -i \\ i & 0 & 0 & 1 \\ 0 & i & 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & -1 & 0 & -i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & -i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & -1 & -i \\ 0 & 0 & i & -1 \end{pmatrix} & \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \end{pmatrix}$$

Now our goal is to determine how many linearly independent matrices there are in this set of nine. We do this by imagining an arbitrary real linear combination of the nine matrices and attempting to solve the system of 16 equations, one for each position in the matrices. The next function creates the matrix corresponding to this system of equations.

```
SystemMatrix[a_, n_] :=
 Table[
     Table[
         Extract[
             Extract[
                 Extract[a, m],
              (j - Mod[j, n]) / n + 1],
             Mod[j, n] + 1],
         {m, 1, Length[a]}
      ],
     {j, 0, n^2 - 1}
 ]
```

Now we define an array of all nine matrices to be passed into the SystemMatrix function.

```
S := {BoxProduct[σ1, σ1], BoxProduct[σ1, σ2], BoxProduct[σ1, σ3],
  BoxProduct[σ2, σ1], BoxProduct[σ2, σ2], BoxProduct[σ2, σ3],
  BoxProduct[σ3, σ1], BoxProduct[σ3, σ2], BoxProduct[σ3, σ3]}

MatrixForm[SystemMatrix[S, 4]]
```

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & -i & 0 & 1 & -i & 0 & 1 & -i & 0 \\ 1 & 1 & 1 & -i & -i & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & i & 0 & 1 & i & 0 & 1 & i & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -i & -i & -i & 0 & 0 & 0 \\ 1 & 1 & 1 & i & i & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & -i & 0 & 1 & -i & 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & i & i & i & 0 & 0 & 0 \\ 1 & i & 0 & 1 & i & 0 & 1 & i & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & -2 \end{pmatrix}$$

```
MatrixForm[TruncateMatrix[RowReduce[SystemMatrix[S, 4]]]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

We have 5 equations in 8 unknowns, so there is definitely a solution, which means the set is linearly dependent. We will try removing four of the matrices.

```
T := {BoxProduct[σ1, σ1], BoxProduct[σ2, σ2],
  BoxProduct[σ1, σ2], BoxProduct[σ2, σ3], BoxProduct[σ3, σ1]}

MatrixForm[TruncateMatrix[RowReduce[SystemMatrix[T, 4]]]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This tells us that the set is linearly independent because the only linear combination that adds to the zero matrix is the one with all coefficients zero. Therefore, there are five linearly indpendent matrices in the original set of nine.

Clebsch–Gordon Relation: 2 x 2 = 3 + 1

Now we want to check if the set of all matrices in the 2x2 representation is the same as the set of all matrices in the direct sum of the matrices in the 3 and 1 representations. We can check this by writing matrices with arbitrary coefficients in front of the generators and checking if the resulting matrices are similar.

```
FullSimplify[CharacteristicPolynomial[BlockMatrix[
   {{MatrixExp[a*J1 + b*J2 + c*J3], ZeroMatrix[3, 1]}, {ZeroMatrix[1, 3], {{d}}}}], x]]
```

$$(-1+x)\,(-d+x)\,\left(1+x^2-2\,x\,\text{Cosh}\left[\sqrt{a^2+b^2+c^2}\,\right]\right)$$

```
FullSimplify[CharacteristicPolynomial[
  MatrixExp[a*BoxProduct[σ1, σ2] + b*BoxProduct[σ2, σ3] + c*BoxProduct[σ3, σ1]], x]]
```

$$(-1+x)^2\,\left(1+x^2-2\,x\,\text{Cosh}\left[2\,\sqrt{a^2+b^2+c^2}\,\right]\right)$$

These two matrices have the same characteristic polynomial (if we set d=1 and multiply a,b,c by a factor of 2 in the first equation), so they are probably similar. See http://planetmath.org/encyclopedia/CharacteristicEquation.html. In conclusion we have found that

$$\{\text{Exp}[a\,\sigma_1\times\sigma_2 \;+\; b\,\sigma_2\times\sigma_3 \;+\; c\,\sigma_3\times\sigma_1] \mid a,b,c \in \mathbb{R}\} = \{\text{BlockDiagonal}[\text{Exp}[\,a\,J_1 + b\,J_2 + c\,J_3],\, I_1] \mid a,b,c \in \mathbb{R}\}$$

where

$$\sigma_i \times \sigma_j \;=\; \sigma_i \otimes I_2 + I_2 \,\otimes\, \sigma_j$$

The question is: Why are there only three parameters in the 2x2 representation when we found that there are five linearly independent matrices?

In Group Theory in Physics by Wu–Ki Tung, page 118 says "The generators of a direct product representation are the sums of the corresponding generators of its constituent representations.

Discussion

It is important to understand the Clebsh–Gordon decomposition because it splits direct product representations into direct sums of irreducible representations, and it is the irreducible representations that show up as independent terms in the Lagrangian. This is why a singlet state can have a much different energy than the triplet states, because they can each have different coefficients on their respective terms in the Lagrangian, but within a multiplet, the symmetry "protects you" from varying the energy (According to Zvi Bern). This all relates to internal symmetries, which are symmetries that mix particle states. The goal is to understand why a symmetry like isospin SU(2), which can be seen in the QCD Lagrangian, causes the states to split into the irreducible representations.

In class, Zvi Bern also mentioned that the gauge bosons are in the adjoint representation and all other particles are in the fundamental representation. I would like to understand what this means. The adjoint representation is the representation with dimension equal to the number of group generators.

Another important fact is that if you use the Kronecker product instead of what I called the box product, then all nine matrices are linearly independent.