

The conserved charge operator corresponding to global gauge symmetry in QED is

$$\hat{Q} = \int \hat{\psi}^\dagger \hat{\psi} d^3x = \int \hat{\psi}_\alpha^* \hat{\psi}_\alpha d^3x$$

We can show that the eigenvalue of this operator is the number of particles minus the number of antiparticles in the eigenstate. This is shown by expressing  $\psi_\alpha$  in terms of creation and annihilation operators. Peskin and Schroeder equation (3.99) says <sup>1</sup>

$$\hat{\psi}_\alpha(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left( \hat{a}_{\mathbf{p}}^s u_\alpha^s(p) e^{-ip \cdot x} + \hat{b}_{\mathbf{p}}^{s*} v_\alpha^s(p) e^{ip \cdot x} \right)$$

Here,  $\alpha$  is the index into the Dirac spinor, and  $s$  is the spin index, which is  $\pm \frac{1}{2}$  for spin 2. Taking the complex conjugate gives

$$\hat{\psi}_\alpha^*(x) = \int \frac{d^3p'}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_{\mathbf{p}'}}} \left( \hat{a}_{\mathbf{p}'}^{r*} u_\alpha^{r*}(p') e^{ip' \cdot x} + \hat{b}_{\mathbf{p}'}^r v_\alpha^{r*}(p') e^{-ip' \cdot x} \right)$$

Now we insert these expressions into the equation for  $\hat{Q}$ .

$$\begin{aligned} \hat{Q} &= \frac{1}{(2\pi)^3} \int d^3x \int d^3p \int d^3p' \frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{p}'}}} \left( \hat{a}_{\mathbf{p}'}^{r*} u_\alpha^{r*}(p') e^{ip' \cdot x} + \hat{b}_{\mathbf{p}'}^r v_\alpha^{r*}(p') e^{-ip' \cdot x} \right) \left( \hat{a}_{\mathbf{p}}^s u_\alpha^s(p) e^{-ip \cdot x} + \hat{b}_{\mathbf{p}}^{s*} v_\alpha^s(p) e^{ip \cdot x} \right) \\ &= \frac{1}{(2\pi)^3} \int d^3x \int d^3p \int d^3p' \frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{p}'}}} \left( \hat{a}_{\mathbf{p}'}^{r*} u_\alpha^{r*}(p') \hat{a}_{\mathbf{p}}^s u_\alpha^s(p) e^{i(p'-p) \cdot x} + \hat{a}_{\mathbf{p}'}^{r*} u_\alpha^{r*}(p') \hat{b}_{\mathbf{p}}^{s*} v_\alpha^s(p) e^{i(p'+p) \cdot x} \right. \\ &\quad \left. + \hat{b}_{\mathbf{p}'}^r v_\alpha^{r*}(p') \hat{a}_{\mathbf{p}}^s u_\alpha^s(p) e^{-i(p'+p) \cdot x} + \hat{b}_{\mathbf{p}'}^r v_\alpha^{r*}(p') \hat{b}_{\mathbf{p}}^{s*} v_\alpha^s(p) e^{-i(p'-p) \cdot x} \right) \end{aligned}$$

In the third term, we swap the dummy indices  $r$  and  $s$ , swap  $p$  and  $p'$ , and apply the anticommutator  $\{a_{\mathbf{p}}^r, b_{\mathbf{p}'}^s\} = 0$ , which negates the term.

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int d^3x \int d^3p \int d^3p' \frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{p}'}}} \left( \hat{a}_{\mathbf{p}'}^{r*} u_\alpha^{r*}(p') \hat{a}_{\mathbf{p}}^s u_\alpha^s(p) e^{i(p'-p) \cdot x} + \hat{a}_{\mathbf{p}'}^{r*} u_\alpha^{r*}(p') \hat{b}_{\mathbf{p}}^{s*} v_\alpha^s(p) e^{i(p'+p) \cdot x} \right. \\ &\quad \left. - \hat{a}_{\mathbf{p}'}^{r*} v_\alpha^{s*}(p) \hat{b}_{\mathbf{p}}^r u_\alpha^r(p') e^{-i(p+p') \cdot x} + \hat{b}_{\mathbf{p}'}^r v_\alpha^{r*}(p') \hat{b}_{\mathbf{p}}^{s*} v_\alpha^s(p) e^{-i(p'-p) \cdot x} \right) \end{aligned}$$

We can now see that the second and third terms are complex conjugates of each other and one is being subtracted, so the difference must be purely imaginary. However, from the original equation for  $\hat{Q}$ , we know that it must be real. Since there is no way for the other terms to cancel the second and third terms because they have different operators, the second and third terms must sum to zero.

$$= \frac{1}{(2\pi)^3} \int d^3x \int d^3p \int d^3p' \frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{p}'}}} \left( \hat{a}_{\mathbf{p}'}^{r*} u_\alpha^{r*}(p') \hat{a}_{\mathbf{p}}^s u_\alpha^s(p) e^{i(p'-p) \cdot x} + \hat{b}_{\mathbf{p}'}^r v_\alpha^{r*}(p') \hat{b}_{\mathbf{p}}^{s*} v_\alpha^s(p) e^{-i(p'-p) \cdot x} \right)$$

Next we perform the integration over  $x$ , which introduces delta functions.

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int d^3p \int d^3p' \frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{p}'}}} \left( \hat{a}_{\mathbf{p}'}^{r*} u_\alpha^{r*}(p') \hat{a}_{\mathbf{p}}^s u_\alpha^s(p) (2\pi)^3 \delta^3(p' - p) + \hat{b}_{\mathbf{p}'}^r v_\alpha^{r*}(p') \hat{b}_{\mathbf{p}}^{s*} v_\alpha^s(p) (2\pi)^3 \delta^3(p - p') \right) \\ &= \int d^3p \frac{1}{2E_{\mathbf{p}}} \left( \hat{a}_{\mathbf{p}}^{r*} u_\alpha^{r*}(p) \hat{a}_{\mathbf{p}}^s u_\alpha^s(p) + \hat{b}_{\mathbf{p}}^r v_\alpha^{r*}(p) \hat{b}_{\mathbf{p}}^{s*} v_\alpha^s(p) \right) \end{aligned}$$

Using the normalization conditions from Peskin and Schroeder (3.60) and (3.63),  $u_\alpha^{r*}(p) u_\alpha^s(p) = 2E_{\mathbf{p}} \delta^{rs}$  and  $v_\alpha^{r*}(p) v_\alpha^s(p) = 2E_{\mathbf{p}} \delta^{rs}$ ,

$$\begin{aligned} &= \int d^3p \frac{1}{2E_{\mathbf{p}}} \left( \hat{a}_{\mathbf{p}}^{r*} \hat{a}_{\mathbf{p}}^s (2E_{\mathbf{p}} \delta^{rs}) + \hat{b}_{\mathbf{p}}^r \hat{b}_{\mathbf{p}}^{s*} (2E_{\mathbf{p}} \delta^{rs}) \right) = \frac{1}{(2\pi)^3} \int d^3p \left( \hat{a}_{\mathbf{p}}^{s*} \hat{a}_{\mathbf{p}}^s + \hat{b}_{\mathbf{p}}^s \hat{b}_{\mathbf{p}}^{s*} \right) \\ &= \int d^3p \left( \hat{a}_{\mathbf{p}}^{s*} \hat{a}_{\mathbf{p}}^s - \hat{b}_{\mathbf{p}}^{s*} \hat{b}_{\mathbf{p}}^s + \infty \right) = \int d^3p \sum_s (N_s^+(p) - N_s^-(p)) + \infty \end{aligned}$$

<sup>1</sup>I think the power of  $2\pi$  is a typo in the book.