Fibonacci Primes

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Definition 0.1. The Fibonacci Sequence is the sequence of positive integers defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$. The sequence begins as follows: $1, 1, 2, 3, 5, 8, 13, 21, 34, ...$

Definition 0.2. $\mathrm{F}_n =$ the n^{th} element of the Fibonacci Sequence

Lemma 0.3. *For any positive integers n and x,* $F_{n+x} = F_{x+1}F_n + F_xF_{n-1}$

Proof. By Definition 0.1, $F_{n+1} = F_n + F_{n-1}$ and $F_{n+2} = F_{n+1} + F_n = 2F_n + F_{n-1}$. F_{n+x} is recursively defined from these first two cases, so we can obtain the coefficients of F_n and F_{n-1} by looking at the elements of the Fibonacci sequence. Observe:

- $F_{n+3} = F_{n+2} + F_{n+1} = 3F_n + 2F_{n-1}$
- $F_{n+4} = F_{n+3} + F_{n+2} = 5F_n + 3F_{n-1}$
- $F_{n+5} = F_{n+4} + F_{n+3} = 8F_n + 5F_{n-1}$
- $F_{n+6} = F_{n+5} + F_{n+4} = 13F_n + 8F_{n-1}$

And by setting the proper starting points, we have $F_{n+x} = F_{x+1}F_n + F_xF_{n-1}$. □

Lemma 0.4. *For any positive integers n and m,* $F_n | F_{nm}$

Proof. We proceed by induction. Fix an integer n and assume that $F_n | F_{nm}$ for all m from 1 to k. Now consider $F_{n(k+1)} = F_{n+nk}$. According to Lemma 0.3, $F_{n+nk} = F_{nk+1}F_n + F_{nk}F_{n-1}$. But by the inductive hypothesis, $F_n | F_{nk}$, so we may write $F_{n+nk} = F_{nk+1}F_n + cF_nF_{n-1} = F_n(F_{nk+1} + cF_{n-1})$. Therefore, $F_n | F_{n+nk} = F_{n(k+1)}$. Since the inductive step holds, and the base case $F_n | F_n$ is obvious, we have that $F_n | F_{nm}$ for all positive integers n and m. □

Theorem 0.5. *If* F_n *is prime, then n is either 4 or prime.*

Proof. By Lemma 0.4, F_n divides F_{nm} for any integers n and m, and thus any element, F_{nm} will definitely be composite whenever $min(F_n, F_m) > 1$, or equivalently, whenever $min(n, m) > 2$, which means mn is a composite integer greater than 4. The only remaining allowable indices for Fibonacci primes are primes and 4. □