Quantum theory in discrete spacetime

Chris Clark
*University of California Los Angeles, Department of Physics and Astronomy, Los Angeles, CA 90095**
(Dated: July 13, 2010)

A discrete spacetime model of the electron field is postulated. It is then argued that the discrete model is observationally consistent with the Dirac equation while providing the theoretical advantage that $\hbar$ can be explained as a measure of the discretization.

PACS numbers: 03.65.Ta

INTRODUCTION

In this paper we will consider the plausibility of the conjecture that spacetime is discrete. Our purpose is not to rigorously prove that spacetime is in fact discrete, but rather to demonstrate results that challenge the belief that spacetime must be continuous. We proceed by postulating a hypothetical discrete spacetime model of the electron field and subsequently showing that we still obtain behavior consistent with observation via the Dirac equation. We will also be able to produce a derivation of the relativistic momentum operator in the position representation without appeal to classical correspondences. Finally, the model will frame $\hbar$ as a measure of the discretization, providing a potential explanation for the value of $\hbar$ and its associated quantum phenomena.

The model

Consider a discrete-time simulation of a set of integer-valued matrices based on dimensionless numerical rules. These matrices are data structures in which each memory slot, containing one matrix element, corresponds to what we think of as a point in space. However, matrices are discrete in the sense that they are indexed by integers, so in this model space is discrete. The values in the matrices correspond to the components of what are commonly referred to as "quantum wave functions" in physics. Of course we usually think of wave functions as containing real-valued components, but if the integer matrix elements span a sufficiently large range, then they will be indistinguishable from real numbers.

For simplicity, we will restrict our model to one dimension. We define the discrete-time simulation on two one-dimensional matrices (arrays), ψ and ρ, according to the following coupled rules of evolution

$$\begin{aligned} \psi'_x &= \psi_x + \rho_x \\ \rho'_x &= \rho_x + (\psi_{x+1} + \psi_{x-1} - 2\psi_x - \mu\psi_x) \end{aligned} \tag{1}$$

where primes indicate the value of the field after one step of evolution, and μ is a dimensionless constant that is characteristic of the particular field under consideration. These rules are completely dimensionless because they merely describe the numerical change in matrix elements based on the numerical value of other matrix elements. However, we also have an interpretation of x as a point in space and the prime as corresponding to a step in time. These dimensional concepts of space and time are based on a perspective coming from inside the simulation, as if the observer were living within the simulation. It is useful to make these rules compatible with the notion of dimensional space and time to make them more easily comparable to physical equations. This can be accomplished by replacing the spacetime increments and decrements with additions and subtractions of δx and δt.

$$\begin{aligned} \psi_x^{(+\delta t)} &= \psi_x + \rho_x \\ \rho_x^{(+\delta t)} &= \rho_x + (\psi_{x+\delta x} + \psi_{x-\delta x} - 2\psi_x - \mu\psi_x) \end{aligned} \tag{2}$$

The steps δx and δt are just equal to 1 from an external point of view, but equal to some dimensionful constant from a point of view coming from within the simulation. Any length in the model universe corresponds to a definite integer number of δx steps and every duration corresponds to a definite integer number of δt steps. So it is possible to assign dimensionful values to these steps, such as some small fraction of a meter and a second. However, these steps are not to be confused with the step sizes in the external universe (the universe that is running the simulation). From within the model universe, there is no way to measure the size of δt_{ext}, which is the time it takes in the external universe for the simulation to evolve by one step.

Discussion of model

These equations (2) represent a decoupling of the discrete Klein-Gordon equation[1], which can be pictured as modeling a chain of oscillators connected by springs with an additional spring at each point that pulls the oscillator back to equilibrium. The ψ_x values represent the amplitudes and the ρ_x values represent the velocities of the oscillators.

There is an important conceptual point to make about simulations in general, which is that information only exists at a particular time if it is stored somewhere at that time. So in the rules for calculating the values in the matrices, there can be no dependence on the field values at times earlier than one step prior. This is why there are no double-primed fields in the rules, and why the fundamental rules of a simulation must always be first-order in time. However, an auxiliary field can be used to store information from the previous instance, which will make the result appear to be second-order in time. This is what we will discover when we derive the Klein-Gordon equation from these rules.

One might ask whether the model really requires two matrices to get the desired results. We can see that one matrix cannot support propagating waves by considering a plane wave in a matrix. The wave is symmetrical in the forward and backward directions, so there is no information available to determine which way the wave is travelling. It is the second matrix, which stores amplitudinal velocity data, that determines the direction.

Notice that if the limit is taken as $\delta t \to 0$ and $\delta x \to 0$, then $\psi_x^{(+\delta t)} \to \psi_x$, so $\rho \to 0$ by the first rule. Similarly in the second rule, $\rho_x^{(+\delta t)} \to \rho_x$, $\psi_{x+\delta x} \to 0$, and $\psi_{x-\delta x} \to 0$, so $\psi \to 0$ since only the $\mu\psi_x$ term remains. Therefore, the only solution is the trivial solution where both fields are zero. This is not a problem because there is no need for the continuum limit to work in our model. Now the steps are completely manifest and we can proceed to show that the Dirac equation holds at sufficiently large scales.

THE KLEIN-GORDON EQUATION

Discrete form

All of the standard equations in physics are based on continuous spacetime, so in order to compare our discrete model to any conventional equations, we will have to approximate the discrete equations with continuous ones. First we will show how the Klein-Gordon equation emerges from our model.

For discrete derivatives, forward and backward differences can be different, unlike continuous derivatives. Therefore we must define the forward and backward discrete derivatives.

$$\frac{\overrightarrow{\delta}\psi_x}{\delta x} \equiv \frac{\psi_{x+\delta x} - \psi_x}{\delta x} \quad \text{and} \quad \frac{\overleftarrow{\delta}\psi_x}{\delta x} \equiv \frac{\psi_x - \psi_{x-\delta x}}{\delta x}$$

Then

$$\begin{aligned}
\frac{\overleftarrow{\delta}}{\delta x}\left(\frac{\overrightarrow{\delta}\psi_x}{\delta x}\right) &= \frac{1}{\delta x}\left(\frac{\overrightarrow{\delta}\psi_x}{\delta x} - \frac{\overrightarrow{\delta}\psi_{x-\delta x}}{\delta x}\right) \\
&= \frac{1}{\delta x}\left(\frac{\psi_{x+\delta x} - \psi_x}{\delta x} - \frac{\psi_x - \psi_{x-\delta x}}{\delta x}\right) \\
&= \frac{1}{\delta x^2}\left(\psi_{x+\delta x} + \psi_{x-\delta x} - 2\psi_x\right)
\end{aligned}$$

So the rules can be re-written as

$$\begin{aligned}
\psi_x^{(+\delta t)} &= \psi_x + \rho_x \\
\rho_x^{(+\delta t)} &= \rho_x + \left(\delta x^2 \frac{\overleftarrow{\delta}}{\delta x}\left(\frac{\overrightarrow{\delta}\psi_x}{\delta x}\right) - \mu\psi_x\right)
\end{aligned}$$

Or if we define $c = \delta x/\delta t$,

$$\begin{aligned}
\psi_x^{(+\delta t)} &= \psi_x + \rho_x \\
\rho_x^{(+\delta t)} &= \rho_x + \left(c^2 \frac{\overleftarrow{\delta}}{\delta x}\left(\frac{\overrightarrow{\delta}\psi_x}{\delta x}\right) - \frac{\mu}{\delta t^2}\psi_x\right)\delta t^2
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\overrightarrow{\delta}^2\psi_x}{\delta t^2} &= \frac{\overrightarrow{\delta}}{\delta t}\left(\frac{\psi_x^{(+\delta t)} - \psi_x}{\delta t}\right) \\
&= \frac{1}{\delta t}\frac{\overrightarrow{\delta}\rho_x}{\delta t} = \frac{1}{\delta t}\frac{\rho_x^{(+\delta t)} - \rho_x}{\delta t}
\end{aligned}$$

We obtain a form of the discrete Klein-Gordon equation

$$\frac{\overrightarrow{\delta}^2\psi_x}{\delta t^2} = c^2 \frac{\overleftarrow{\delta}}{\delta x}\left(\frac{\overrightarrow{\delta}\psi_x}{\delta x}\right) - \frac{\mu}{\delta t^2}\psi_x \qquad (3)$$

Continuous form

Now if δx and δt are small compared to our most precise measurement scales, then it would not be possible to resolve all of the detail in the function ψ_x. We would only be able to measure the average value of ψ_x over some large number of steps in both x and t. So we can define the effective wave function $\tilde{\psi}_x$ to be the moving average of ψ_x over the number of steps corresponding to our most precise measurement. This has the effect of smoothing out the oscillations in ψ_x, making it approximately linear on the scale of a step size.

Now if we represent ψ_x as a Fourier-like series of monochromatic solutions to the equation, then $\tilde{\psi}_x$ is equal to this Fourier series with the high-frequency solutions truncated off by the smoothing procedure. Therefore $\tilde{\psi}_x$ will also obey the equation since it is just a linear combination of solutions to a linear equation.

So $\tilde{\psi}_x$ is a solution to the same equation as ψ_x, which is indistinguishable from ψ_x, but smoothed out so much that it is almost exactly linear on the scale of a few steps in x or t. Since it is so close to linear, the discrete derivative is almost exactly equal to the slope, so we can replace $\tilde{\psi}_x$ with a continuous function $\psi(x,t)$ and replace the discrete derivatives with continuous derivatives without introducing a measurable difference.

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2} - \frac{\mu}{\delta t^2}\psi \qquad (4)$$

We must stress that we did not take the limit as δx and δt go to zero or change their sizes in any way—they are still equal to one. All we have done is found an approximate equation for a continuous effective wave function that captures all of the measurable information that is in the discrete wave function. This equation is very similar to the Klein-Gordon equation; the only difference is the last term. To make the equations match, we define a constant called $\hbar$ to be

$$\hbar \equiv \frac{mc^2 \delta t}{\sqrt{\mu}} \tag{5}$$

and assume that μ is proportional to m^2 so that $\hbar$ takes on the same value for all types of particles. Then we obtain the Klein-Gordon equation

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2} - \frac{m^2 c^4}{\hbar^2} \psi \tag{6}$$

The fact that $\hbar$ is proportional to δt means that $\hbar$ is a measure of the discretization in this model. So in this model, $\hbar$ is not an arbitrary constant, but a well-understood consequence of discrete spacetime.

THE DIRAC EQUATION

Spinors

The Dirac equation is an automatic consequence of assuming that a system obeying the Klein-Gordon equation is actually governed by an equation that is first-order in space and time-derivatives. The original rules we defined were first-order in time derivatives, so the approximate continuous equation would be the Dirac equation. To get the one-dimensional Dirac equation, we may factor the Klein-Gordon equation to get back to a set of differential equations that are first-order in time. Any equation that is n^{th} order in time derivatives can be split into a system of n equations that are first-order in time derivatives using one simple trick: define the fields $\phi_1 = \frac{\partial \psi}{\partial t}, \phi_2 = \frac{\partial \phi_1}{\partial t}, ...$, each of these counting as one first-order equation, and insert the appropriate ϕ_i for all higher-order derivatives in the original equation. However, we also want to be able to write the set of differential equations as one vector equation. This is the motivation for the concept of spinors: one n-dimensional vector field obeying an equation that is first-order in time can be used to represent a single scalar field obeying an equation that is n^{th} order in time.

Converting to spinor form

The spinor equation for the rules will be simplest when the two equations are as symmetrical as possible. So rather than simply defining $\phi = \frac{\partial \psi}{\partial t}$, we introduce the symmetrical field ϕ to substitute for ρ and postulate

$$\frac{\partial \psi}{\partial t} = \hat{\alpha} \phi \quad \text{and} \quad \frac{\partial \phi}{\partial t} = \hat{\beta} \psi$$

This decomposition is not the only way to reach the Dirac equation, but it is the most intuitive. Now assuming $\hat{\alpha}$ and $\hat{\beta}$ have no explicit time dependence,

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t} (\hat{\alpha} \phi) = \hat{\alpha} \hat{\beta} \psi$$

Therefore,

$$\hat{\alpha} \hat{\beta} = c^2 \frac{\partial^2}{\partial x^2} - \frac{m^2 c^4}{\hbar^2}$$

There are many ways to factor out $\hat{\alpha}$ and $\hat{\beta}$. Each non-trivial choice gives a different possible representation of the Dirac equation. We choose

$$\hat{\alpha} = -c \frac{\partial}{\partial x} + \frac{mc^2}{\hbar} \quad \text{and} \quad \hat{\beta} = -c \frac{\partial}{\partial x} - \frac{mc^2}{\hbar}$$

so that

$$\frac{\partial \psi}{\partial t} = \left(-c \frac{\partial}{\partial x} + \frac{mc^2}{\hbar} \right) \phi \quad \text{and} \quad \frac{\partial \phi}{\partial t} = \left(-c \frac{\partial}{\partial x} - \frac{mc^2}{\hbar} \right) \psi$$

These equations can be combined into the vector equation

$$\frac{\partial}{\partial t} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = -c \frac{\partial}{\partial x} \begin{pmatrix} \phi \\ \psi \end{pmatrix} + \frac{mc^2}{\hbar} \begin{pmatrix} \phi \\ -\psi \end{pmatrix}$$

$$= -c \frac{\partial}{\partial x} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} + \frac{mc^2}{\hbar} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

We can now define a spinor $\chi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$ so that

$$\frac{\partial \chi}{\partial t} = \left(-c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \frac{mc^2}{\hbar} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \chi$$

Multiplying by $i\hbar$ and using the Pauli matrices,

$$i\hbar \frac{\partial \chi}{\partial t} = \left(-i\hbar c \sigma_x \frac{\partial}{\partial x} - mc^2 \sigma_y \right) \chi \tag{7}$$

This is a representation of the one-dimensional Dirac equation.[2]

THE MOMENTUM OPERATOR

State-vector formalism

Let's now return to the original discrete rules. So far we have been working with wave functions, but quantum theory is usually done in a state vector formalism in which each field configuration corresponds to a vector in a one-to-one fashion. Let $|x\rangle$ be the state vector corresponding to the field for which $\psi_{x'} = \delta_{x'x}$, where δ here is the Kronecker delta. The set of all $|x\rangle$ defines an orthonormal basis for the space of all state vectors, which we call the position basis. The field ψ is thus represented by the state vector $|\psi\rangle = \sum_x \psi_x |x\rangle$. Operators act on state vectors to extract information and/or modify the state. The position operator $\hat{x}$ is a Hermitian operator defined by $\hat{x}|x\rangle = x|x\rangle$ which extracts position information. The spatial translation operator $\hat{T}(\Delta x)$ shifts the values of the field for a state and is defined by $\hat{T}(\Delta x)|x\rangle = |x+\Delta x\rangle$ for any length $\Delta x = n\delta x$, where n is an integer. Shifting preserves the norm of the vector, so if $|\psi'\rangle = \hat{T}(\Delta T)|\psi\rangle$, we must have $\langle\psi'|\,\psi'\rangle = \left\langle\psi\middle|\hat{T}^\dagger(\Delta x)\hat{T}(\Delta x)\middle|\psi\right\rangle = \langle\psi|\,\psi\rangle$, which means that $T^\dagger(\Delta x) = T^{-1}(\Delta x) = T(-\Delta x)$. From the definition, we also see that

$$
\begin{aligned}
[\hat{x}, \hat{T}(\Delta x)]\,|x\rangle &= \hat{x}\hat{T}(\Delta x)\,|x\rangle - \hat{T}(\Delta x)\hat{x}\,|x\rangle \\
&= \hat{x}\,|x+\Delta x\rangle - x\hat{T}(\delta x)\,|x\rangle \\
&= (x+\Delta x)\,|x+\Delta x\rangle - x\,|x+\Delta x\rangle \\
&= \Delta x\hat{T}(\Delta x)\,|x\rangle
\end{aligned}
$$

So $[\hat{x}, \hat{T}(\Delta x)] = \Delta x\hat{T}(\Delta x)$ on position eigenstates $|x\rangle$, but since these position eigenstates form a basis, this relation holds for all state vectors.

Deriving the momentum operator

We now define the operator

$$\hat{p} \equiv \frac{1}{2c^2}\left(\hat{v}\hat{H} + \hat{H}\hat{v}\right) \tag{8}$$

where $\hat{H}$ is defined by $\hat{H}\psi_x = i\hbar\frac{\overrightarrow{\delta}\psi_x}{\delta t}$ and $\hat{v}$ is the velocity operator $\frac{1}{i\hbar}[\hat{x}, \hat{H}]$, coming from Ehrenfest's theorem. We can simplify to gain some insight into the operator.

$$
\begin{aligned}
\hat{p} &= \frac{1}{2i\hbar c^2}\left([\hat{x},\hat{H}]\hat{H} + \hat{H}[\hat{x},\hat{H}]\right) \\
&= \frac{1}{2i\hbar c^2}\left(\hat{x}\hat{H}^2 - \hat{H}\hat{x}\hat{H} + \hat{H}\hat{x}\hat{H} - \hat{H}^2\hat{x}\right) \\
&= \frac{1}{2i\hbar c^2}[\hat{x},\hat{H}^2] = \frac{1}{2i\hbar c^2}\left[\hat{x}, -\hbar^2\frac{\overrightarrow{\delta}^2}{\delta t^2}\right] \\
&= \frac{i\hbar}{2c^2}\left[\hat{x}, c^2\frac{\overleftarrow{\delta}}{\delta x}\frac{\overrightarrow{\delta}}{\delta x} - \frac{m^2c^4}{\hbar^2}\right] \\
&= \frac{i\hbar}{2}\left[\hat{x}, \frac{1}{\delta x^2}\left(\hat{T}(\delta x) + \hat{T}(-\delta x) - 2\hat{I}\right)\right] \\
&= \frac{i\hbar}{2\delta x^2}\left([\hat{x},\hat{T}(\delta x)] + [\hat{x},\hat{T}(-\delta x)]\right) \\
&= \frac{i\hbar}{2\delta x^2}\left((\delta x)\hat{T}(\delta x) + (-\delta x)\hat{T}(-\delta x)\right) \\
&= \frac{i\hbar}{2\delta x}\left(\hat{T}(\delta x) - \hat{T}(-\delta x)\right)
\end{aligned}
$$

Therefore, the position representation of the momentum operator is

$$
\begin{aligned}
\langle x|\,\hat{p}\,|\psi\rangle &= \frac{i\hbar}{2\delta x}\left\langle x\middle|\hat{T}(\delta x) - \hat{T}(-\delta x)\middle|\psi\right\rangle \\
&= \frac{i\hbar}{2\delta x}\left(\langle x-\delta x|\,\psi\rangle - \langle x+\delta x|\,\psi\rangle\right) \\
&= \frac{i\hbar}{2\delta x}\left(\psi_{x-\delta x} - \psi_{x+\delta x}\right)
\end{aligned}
$$

So

$$\langle x|\,\hat{p}\,|\psi\rangle = -i\hbar\frac{\overleftrightarrow{\delta}\psi_x}{\delta x} \tag{9}$$

where we have defined the operator

$$\frac{\overleftrightarrow{\delta}\psi_x}{\delta x} \equiv \frac{1}{2}\left(\frac{\overrightarrow{\delta}\psi_x}{\delta x} + \frac{\overleftarrow{\delta}\psi_x}{\delta x}\right) = \frac{\psi_{x+\delta x} - \psi_{x-\delta x}}{2\delta x}$$

This shows that $\hat{p}$ has the expected form for the quantum momentum operator, but does it actually represent the momentum? We can check by examining the classical limit when all quantities can be measured simultaneously. First recall the Klein-Gordon equation derived earlier.

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2} - \frac{m^2c^4}{\hbar^2}\psi$$

In the classical limit, the momentum operator in the position representation derived above is approximated by $\hat{p} = -i\hbar\frac{\partial}{\partial x}$, which means $\hat{p}^2 = -\hbar^2\frac{\partial^2}{\partial x^2}$. Inserting this expression into the Klein-Gordon equation,

$$\frac{\partial^2\psi}{\partial t^2} = \left(-\frac{c^2}{\hbar^2}\hat{p}^2 - \frac{m^2c^4}{\hbar^2}\right)\psi$$

Now the definition of $\hat{H}$ indicates that in the classical limit, $\hat{H}^2 = -\hbar^2\frac{\partial^2}{\partial t^2}$, so

$$\hat{H}^2 = c^2\hat{p}^2 + m^2c^4$$

So if we consider the classical limit where all states are eigenstates of every operator,

$$\hat{p} = \frac{1}{2c^2}\left(\hat{v}\hat{H} + \hat{H}\hat{v}\right) \quad \rightarrow \quad p = \frac{1}{c^2} vH$$

Squaring,

$$p^2 = v^2 H^2/c^4 = v^2(p^2c^2 + m^2c^4)/c^4$$

Solving for p^2,

$$p^2 = \frac{m^2v^2}{1 - v^2/c^2}$$

Upon taking the square root and inserting the relativistic gamma factor,

$$p = \pm\gamma m v \tag{10}$$

This is the expected expression for the relativistic momentum, so we have shown that the operator for relativistic momentum in the position representation for this model is $-i\hbar\frac{\overleftrightarrow{\delta}}{\delta x}$. The conventional method for deriving this expression relies on the assumption that momentum is the generator of spatial translation or that the canonical commutation relation is $[\hat{x}, \hat{p}] = i\hbar$. In this analysis, we never had to make this assumption; the only assumption was the form of the evolution rules in equation (1). This analysis can also be carried out with continuous operators if the continuous Klein-Gordon equation is used as a starting point as is shown in Appendix A.

LENGTH CONTRACTION

It is possible to show that length contraction occurs as a result of motion in this model universe. Here we will restrict ourselves to the case of solutions that are stationary or rigidly translating so that all time dependence is due to motion. Consider again the Klein-Gordon equation

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2} - \frac{m^2c^4}{\hbar^2}\psi$$

A stationary solution $\phi(u)$ will satisfy

$$0 = c^2\frac{\partial^2\phi}{\partial u^2} - \frac{m^2c^4}{\hbar^2}\phi$$

where we have replaced the parameter x with u for clarity. We want to determine what a stationary solution would look like if it was moving at velocity v. If the moving solution ψ was the same shape and size as the stationary solution ϕ, then the relationship would be $\psi(x,t) = \phi(u(x,t))$ where $u(x,t) = x - vt$. But this does not solve the Klein-Gordon equation. Therefore we will make the Ansatz that the moving solution is length-contracted by some undetermined factor γ_v. A length-contracted stationary state can be written as

$$\phi_{\gamma_v}(u) = \phi_0(\gamma_v u)$$

where ϕ_0 is the uncontracted form of the same function. So we can express our Ansatz as

$$\psi(x,t) = \phi_{\gamma_v}(u_0(x,t)) \quad \text{where} \quad u_0(x,t) = x - vt$$

Simplifying,

$$\psi(x,t) = \phi_0(u(x,t)) \quad \text{where} \quad u(x,t) = \gamma_v(x - vt)$$

Then we find by the chain rule,

$$\begin{aligned}
\frac{\partial\psi}{\partial t} &= \frac{\partial\phi_0}{\partial u}\frac{\partial u}{\partial t} = -v\gamma_v\frac{\partial\phi_0}{\partial u} \\
\frac{\partial^2\psi}{\partial t^2} &= -v\gamma_v\frac{\partial}{\partial u}\left(\frac{\partial\phi_0}{\partial t}\right) = v^2\gamma_v^2\frac{\partial^2\phi_0}{\partial u^2} \\
\frac{\partial\psi}{\partial x} &= \frac{\partial\phi_0}{\partial u}\frac{\partial u}{\partial x} = \gamma_v\frac{\partial\phi_0}{\partial u} \\
\frac{\partial^2\psi}{\partial x^2} &= \gamma_v\frac{\partial}{\partial u}\left(\frac{\partial\phi_0}{\partial x}\right) = \gamma_v^2\frac{\partial^2\phi_0}{\partial u^2}
\end{aligned}$$

So if ψ is a solution to the time-dependent Klein-Gordon equation, we must have

$$v^2\gamma_v^2\frac{\partial^2\phi_0}{\partial u^2} = c^2\gamma_v^2\frac{\partial^2\phi_0}{\partial u^2} - \frac{m^2c^4}{\hbar^2}\phi_0$$

Or equivalently,

$$0 = (c^2 - v^2)\gamma_v^2\frac{\partial^2\phi_0}{\partial u^2} - \frac{m^2c^4}{\hbar^2}\phi_0$$

This we recognize to be the time-independent equation when $(c^2 - v^2)\gamma_v^2 = c^2$, which we know to be true since ϕ_0 is a solution to this equation by its definition. Solving this equation for γ_v gives

$$\gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Thus we have derived the relativistic length contraction factor and validated our assumption that length contraction occurs in this model universe, at least for the class of solutions that we considered. In this model, it is not space that contracts, but the quantum wave functions themselves. This picture is known as neo-Lorentzian ether theory.

CONCLUSIONS

The fact that we were able to derive the relativistic Dirac equation, the relativistic momentum operator, and length contraction of stationary states from a discrete lattice theory is strong evidence that discrete models of the universe are not in contradiction with special relativity as it may initially seem.[3] Furthermore, there are benefits of the discrete model over the continuous model including the ability to provide an explanation for the constant $\hbar$.

I would like to thank Andrew Forrester for finding a critical error in an early draft of this paper. I would also like to thank Yu Guo and William Newman for valuable advice during the preparation of this paper.

APPENDIX A

The derivation for the momentum operator in the position representation also works for continuous operators. Starting from the result

$$\hat{p} = \frac{1}{2i\hbar c^2}\left[\hat{x}, \hat{H}^2\right]$$

and applying $\hat{p}$ to a state ψ that is a solution of the continuous Klein-Gordon equation we find:

$$\begin{aligned}
\hat{p}\psi &= \frac{1}{2i\hbar c^2}\left[\hat{x}, -\hbar^2\frac{\partial^2}{\partial t^2}\right]\psi \\
&= \frac{i\hbar}{2c^2}\left[\hat{x}, c^2\frac{\partial^2}{\partial x^2} - \frac{m^2c^4}{\hbar^2}\right]\psi \\
&= \frac{i\hbar}{2}\left[\hat{x}, \frac{\partial^2}{\partial x^2}\right]\psi \\
&= \frac{i\hbar}{2}\left(x\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2}{\partial x^2}(x\psi)\right) \\
&= \frac{i\hbar}{2}\left(x\frac{\partial^2\psi}{\partial x^2} - \frac{\partial}{\partial x}\left(\psi + x\frac{\partial\psi}{\partial x}\right)\right) \\
&= \frac{i\hbar}{2}\left(x\frac{\partial^2\psi}{\partial x^2} - \frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial x} - x\frac{\partial^2\psi}{\partial x^2}\right) \\
&= \frac{i\hbar}{2}\left(-2\frac{\partial\psi}{\partial x}\right) \\
&= -i\hbar\frac{\partial\psi}{\partial x}
\end{aligned}$$

* URL: `http://www.dfcd.net`

[1] Karachalios N., “Global existence in infinite lattices of nonlinear oscillators: The Discrete Klein-Gordon equation,” Glasgow Math. J. **48**, 463 (2006). Equation (1) is the general form of the Discrete Klein-Gordon equation.

[2] Gaveau B., Jacobson T., Kac M. and Schulman L. S., “Relativistic Extension of the Analogy between Quantum Mechanics and Brownian Motion,” Phys. Rev. Lett. **53**, 419 (1984). Equation (4) is the 1-D Dirac equation.

[3] Forrest, P., “Is space-time discrete or continuous? — An empirical question,” Synthese **103** (3), 327 (1995).