Units and Dimensions

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Introduction

Whenever we make a measurement we obtain a number. For example, by measuring the length of an object with a ruler, we find the number of tick marks between the two ends of the object. Now this number has some attached meaning. We cannot add it to the number of ticks of a clock and get a meaningful result, but we can add it to the number of tick marks from another object to get the length of the two objects end to end. So when we make a measurement, we don't call the result a number, we call it a **quantity**. A quantity is an expression containing a number plus an additional piece of information called the *units*. A **unit** is a standard quantity which serves as a reference from which all other quantities can be obtained by applying scale factors. The standard quantity is completely arbitrary, so it is usually chosen for convenience. In different situations, there may be different units which are more convenient, so there may be several distinct units that can interchangeably describe the same quantity. For this reason we introduce the concept of **dimension**, which assigns a single name to each type of measurement, though different types of measurements can have the same dimension. It is possible for a unit to have no dimension, the most common example being angular measurements. Angles are described by a ratio of arc-length to radius, which causes the dimension of length to cancel, but the radian and degree are standard quantities, which makes them units.

The notation established by James Clerk Maxwell in the 19th century expresses the quantity X as $X = \{X\}[X]$ where $\{X\}$ is the numerical value and $[X]$ represents the units of the quantity X. The dimension of a unit U will be written $[U]$, and thus the dimensions of a quantity X will be written $[[X]]$.

A **system of units** is a set of rules and conventions used to facilitate the reporting of measurements. Any system of units must satisfy two basic requirements. First, it must be possible to represent the result of any physical measurement within the system of units. Second, upon rescaling any particular unit, all equations between measured quantities must remain true.

When forming equations, measured quantities can be added or subtracted only if they have the same dimensions. However, measured quantities can be multiplied or divided regardless of their dimensions, with the result having dimensions given by the product or quotient of the original dimensions. This means that dimensions can sometimes be broken down into factors, which suggests the existence of **fundamental dimensions** that cannot be further decomposed. The problem is that there is no mathematical way to determine which are the fundamental dimensions. Given any set, you can always replace any element with itself times one of the other elements and still have a set of fundamental dimensions. Therefore we will have to use our intuition to determine the set which is conceptually simplest. The answer is length, mass, time, charge, and count. [1] To illustrate the ambiguity, consider the MKSA system in which charge is replaced by current. Current is charge/time, so you can still get the dimension of charge by multiplying current by time. Mathematically this system works just as well, but conceptually it is more complicated because charge is an intrinsic property of matter like mass, whereas current can only be meaningful if a finite duration of time is considered. This additional requirement makes current less fundamental than charge. The only reason the MKSA system uses current is because it is experimentally easier to measure current than it is to measure charge, but a system of units should not depend on the arbitrary limits of modern technology. A **fundamental unit** is a unit that measures a fundamental dimension. The base units of the SI system are the meter, kilogram, second, coulomb, and mole. [2]

[1] Temperature is often considered a fundamental dimension, but it can be expressed in terms of energy, which can be generated from these.

[2] This is the author's view, which is not generally accepted.

Constraints

The number of fundamental dimensions in the system of units described above is five. We will call any system of units with five fundamental dimensions a **physical system of units**. This term is useful because there are many systems of units which have fewer fundamental dimensions. This feat is accomplished by imposing a constraint between two or more base units. Each constraint will reduce the number of fundamental dimensions by 1. Mathematically, a constraint makes the claim that some fundamental dimension is a linear combination of the others. For example, probably the most common constraint is $c = 1$. In MKSA, this says $\{c\}\frac{m}{s} = 1$ so $1s = \{c\}m$, which means that a second is equal to some number of meters. Now this is completely different from the standard use of the term light-second; the dimension of a light-second is still length. Here we are actually saying that a light-second is equal to a second, in other words, seconds and meters are proportional. [3]

Now it is not necessary to believe that this is true in order to use it. You can always use such a convention for shorthand as long as you understand what it means. It does not make sense to truly think that meters and seconds are proportional. We can tell that meters in one spatial direction are indistinguishable from meters in a another spatial dimension because we can simply rotate a meter stick from one direction to another. However, we can not rotate a meter stick into the time dimension, so we have no empirical reason to suspect that time can be measured in meters.

Information Loss

Any constrained system of units loses information about the dimension of the quantity that can only be reconstructed with the knowledge of what the proper units should be. An unconstrained system of units will always present all information about the dimension of any quantity. In practice there is no information lost in either case because the user will always know what the proper dimension is supposed to be, but it is best for the system of units to take on the responsibility.

Lets consider natural units with $c = \hbar = 1$. Then we have

$$c = 1 \Rightarrow [[c]] = [[1]] \Rightarrow \frac{L}{T} = 1 \Rightarrow L = T$$

$$\hbar = 1 \Rightarrow [[\hbar]] = [[1]] \Rightarrow \frac{ML^2}{T} = 1 \Rightarrow M = \frac{1}{L}$$

$$E = \frac{ML^2}{T^2} = M = \frac{1}{L} = \frac{1}{T}$$

Therefore M, L, and T can all be expressed in terms of energy or any other single dimension. This is why information is lost. For example, consider the SI quantity

$$X_{SI} = 23\ kg^2 m/s^2$$

We can convert this into natural units using $\{c\}\frac{m}{s} = 1$, $\{\hbar\}\frac{kg\,m^2}{s} = 1$, and the operation of multiplying both sides of the equation by powers of 1.

$$\rightarrow 23\ kg^2\{c\}^3 m^4/s^5$$

$$\rightarrow 23\ \{c\}^3/(\{\hbar\}^2 s^3)$$

$$X_{NU} = 5.5728 \times 10^{94}\ s^{-3}$$

If this last expression was all you could see, then you would not know much about the real dimension of X. Notice that we cannot go backwards to retrieve the expression X_{SI}, so we have executed a one-way process, indicating the information has been lost. Natural units are not just the result of choosing a different set of base units, as some people have suggested. In general, the more fundamental dimensions there are in use, the more work the system of units does for you. In fact, in some problems it is helpful to assign different fundamental dimensions to different spatial directions. The author believes that the savings in brevity given by the use of constraints do not outweigh the costs in loss of power and clarity.

[3] This should not be confused with setting $\{c\} = 1$, which is not a constraint, but a rescaling of the units of length or time or both.