Fourier Integral Representation of the Dirac Delta Function

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The Problem

It is often claimed in the physics literature that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\ dk$ is equal to the Dirac delta function, but this relation is not strictly true because the integral is not convergent. To see why this integral cannot converge, just consider the real part, which is $\frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(kx)\ dk$ by Euler's formula. The integral is oscillatory, so it never converges to an exact value. However, it is still possible to define a correspondence based on a double integral. Let the function $g(x)$ correspond to the integral $\int_{-\infty}^{\infty} f(x,k)\ dk$ iff

$$PV \int_{-\infty}^{\infty} \int_{a}^{b} f(x,k)\ dx\ dk = \int_{a}^{b} g(x)\ dx$$

for all finite choices of a, b. Here, PV denotes the principal value of the integral. Notice that the order of integration has been switched. Switching the order of integration back is not justified and will break the equality. If we write this correspondence as $\int_{-\infty}^{\infty} f(x,k)\ dk \doteq g(x)$, then we can show that

$$\int_{-\infty}^{\infty} e^{ikx}\ dk \doteq 2\pi\delta(x)$$

The Correspondence

Theorem 1. $\int_{-\infty}^{\infty} e^{ikx}\ dx \doteq 2\pi\delta(x)$

Proof. We must show

$$PV \int_{-\infty}^{\infty} \int_{a}^{b} e^{ikx}\ dx\ dk = \int_{a}^{b} 2\pi\delta(x)\ dx$$

for all finite values of a, b.

$$PV \int_{-\infty}^{\infty} \int_{a}^{b} e^{ikx}\ dx\ dk = PV \int_{-\infty}^{0} \int_{a}^{b} e^{ikx}\ dx\ dk + PV \int_{0}^{\infty} \int_{a}^{b} e^{ikx}\ dx\ dk$$

Performing the substitution $k \to -k$ in the first integral,

$$= PV \int_{\infty}^{0} \int_{a}^{b} e^{i(-k)x}\ dx\ d(-k) + PV \int_{0}^{\infty} \int_{a}^{b} e^{ikx}\ dx\ dk$$

$$= PV \int_{0}^{\infty} \int_{a}^{b} e^{-ikx}\ dx\ dk + PV \int_{0}^{\infty} \int_{a}^{b} e^{ikx}\ dx\ dk$$

$$= 2PV \int_{0}^{\infty} \int_{a}^{b} \mathrm{Re}[e^{ikx}]\ dx\ dk$$

Using Euler's formula, $e^{ia} = \cos(a) + i\sin(a)$,

$$= 2PV \int_{0}^{\infty} \int_{a}^{b} \cos(kx)\ dx\ dk$$

$$= 2 \lim_{\epsilon\to 0} \lim_{N\to\infty} \int_{\epsilon}^{N} \int_{a}^{b} \cos(kx)\ dx\ dk$$

$$= 2 \lim_{\epsilon\to 0} \lim_{N\to\infty} \int_{\epsilon}^{N} \left(\frac{\sin(bk)}{k} - \frac{\sin(ak)}{k}\right)\ dk$$

Since the lower limit of integration never reaches zero, we do not have to worry about the division by zero.

$$= 2 \lim_{\epsilon \to 0} \lim_{N \to \infty} \left(\int_{\epsilon}^{N} \frac{\sin(bk)}{k} \ dk - \int_{\epsilon}^{N} \frac{\sin(ak)}{k} \ dk \right)$$

Now we do the substitution $u = bk$ in the first integral and $u = ak$ in the second.

$$= 2 \lim_{\epsilon \to 0} \lim_{N \to \infty} \left(\int_{b\epsilon}^{bN} \frac{\sin(u)}{u} \ du - \int_{a\epsilon}^{aN} \frac{\sin(u)}{u} \ du \right)$$

If either of the variables a and b are negative, the limits become 0 to $-\infty$, which negates the value of the integral since the integrand is symmetric about zero. Therefore,

$$= 2 \lim_{\epsilon \to 0} \lim_{N \to \infty} \left(\operatorname{sgn}(b) \int_{\epsilon}^{N} \frac{\sin(u)}{u} \ du - \operatorname{sgn}(a) \int_{\epsilon}^{N} \frac{\sin(u)}{u} \ du \right)$$

The integral here is called the sine integral where the upper limit of integration is the parameter to the sine integral function.

$$= 2(\operatorname{sgn}(b) - \operatorname{sgn}(a))\operatorname{Si}(\infty) = \pi(\operatorname{sgn}(b) - \operatorname{sgn}(a))$$

since $\operatorname{Si}(\infty) = \pi/2$. Now, we notice that $\int_a^b 2\pi\delta(x) \ dx = \pi(\operatorname{sgn}(b) - \operatorname{sgn}(a))$, which completes the proof. □

The Fourier Transform

As in illustration of the usefulness of this correspondence, we will (non-rigorously) derive the expression for the Fourier transform by assuming that the correspondence behaves like an equality when double integrals are present, and assuming that the order of integration is interchangeable. Starting with the sifting property of the Dirac delta function,

$$\psi(x) = \int_{-\infty}^{\infty} \psi(x')\delta(x - x') \ dx'$$

$$= \int_{-\infty}^{\infty} \psi(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} \ dk \right] \ dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(t)e^{-ikx'} \ dx' \right] e^{ikx} \ dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)e^{ikx} \ dk$$

where

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x')e^{-ikx'} \ dx'$$

is the Fourier transform of $\psi(x)$.